

HYDRODYNAMIC LIMIT FOR THE GINZBURG-LANDAU $\nabla\phi$ INTERFACE MODEL WITH A CONSERVATION LAW AND THE DIRICHLET BOUNDARY CONDITIONS

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ABSTRACT. Hydrodynamic limit for the Ginzburg-Landau $\nabla\phi$ interface model with a conservation law was established in [6] under the periodic boundary conditions. This paper studies the same problem on the bounded domain imposing Dirichlet boundary conditions. A nonlinear partial equation of fourth order with boundary conditions is derived as the macroscopic equation, which is related to the Wulff shape derived by [2].

1. INTRODUCTION

The Ginzburg-Landau $\nabla\phi$ interface model determines stochastic dynamics for a discretized hypersurface separating two microscopic phases embedded in the $d+1$ dimensional space. The position of the hypersurface is described by height variables $\phi = \{\phi(x); x \in \Gamma\}$ measured from a fixed d -dimensional discrete hyperplane Γ . We will take $\Gamma = \Gamma_N := (\mathbb{Z}/N\mathbb{Z})^d$ when we consider the system on a discretized torus with periodic boundary condition, or $\Gamma = D_N \subset \mathbb{Z}^d$ when we consider the system on the domain with some boundary condition. Here, D_N is a microscopic domain corresponding to a given macroscopic domain $D \subset \mathbb{R}^d$ which is bounded and has a smooth boundary. We then admit an energy (Hamiltonian) for the interface ϕ by

$$H(\phi) = \frac{1}{2} \sum_{\substack{x, y \in \Gamma, \\ |x-y|=1}} V(\phi(x) - \phi(y)) + \sum_{\substack{x \in \Gamma, y \in \mathbb{Z}^d \setminus \Gamma, \\ |x-y|=1}} V(\phi(x) - \phi(y))$$

with a potential $V \in C^2(\mathbb{R})$. Note that we need to give a boundary condition $\{\phi(x); x \in \mathbb{Z}^d \setminus D_N\}$ in order to define the Hamiltonian H . Once we introduce the Hamiltonian H , the dynamics of the interface can be introduced by means of the Langevin equation

$$d\phi_t(x) = -U_x(\phi_t) dt + \sqrt{2}dw_t(x), \quad x \in \Gamma, \quad (1.1)$$

where $\{w_t(x); x \in \Gamma\}$ is a family of independent copies of the one dimensional standard Brownian motion, and $U_x(\phi)$ is defined by

$$U_x(\phi) := \frac{\partial H}{\partial \phi(x)}(\phi) \equiv \sum_{y \in \mathbb{Z}^d; |x-y|=1} V'(\phi(x) - \phi(y))$$

The hydrodynamic scaling limit has been established for the dynamics governed by (1.1) with $\Gamma = \Gamma_N$ in [5], for the dynamics with $\Gamma = D_N$ and the Dirichlet boundary condition in

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[7]. In both cases, the macroscopic motion is described by the nonlinear partial differential equation

$$\begin{aligned} \frac{\partial}{\partial t} h(t, \theta) &= \operatorname{div} \left\{ (\nabla \sigma)(\nabla h(t, \theta)) \right\} \\ &\equiv \sum_{i=1}^d \frac{\partial}{\partial \theta_i} \left\{ \frac{\partial \sigma}{\partial u_i} (\nabla h(t, \theta)) \right\}, \quad \theta \in D, t > 0 \end{aligned}$$

with an appropriate boundary condition.

The dynamics (1.1) can be regarded as the model corresponding to the Glauber dynamics in the particles' systems. Let us introduce the model corresponding to the Kawasaki dynamics in the particles' systems as follows:

$$d\phi_t(x) = -(-\Delta_\Gamma)U(\phi_t) dt + \sqrt{2}d\tilde{w}_t(x), \quad x \in \Gamma, \quad (1.2)$$

where Δ_Γ is the discrete Laplacian on Γ defined by

$$\begin{aligned} (\Delta_\Gamma \psi)(x) &\equiv \sum_{y \in \Gamma} \Delta_\Gamma(x, y) \psi(y) \\ &= \sum_{y \in \Gamma, |x-y|=1} \{\psi(y) - \psi(x)\}, \quad \psi \in \mathbb{R}^\Gamma, x \in \Gamma, \end{aligned}$$

and $\{\tilde{w}_t(x); x \in \Gamma\}$ is a family of Gaussian processes with mean zero and covariance structure

$$E[\tilde{w}_t(x)\tilde{w}_t(y)] = -\Delta_\Gamma(x, y)t \wedge s, \quad x, y \in \Gamma, t, s \geq 0.$$

We note that the dynamics (1.2) preserves the sum $\sum_{x \in \Gamma} \phi_t(x)$, which can be regarded as the volume of the phase under the interface.

The main purpose in this paper is to establish the hydrodynamic scaling limit of ϕ_t determined by (1.2) under the Dirichlet boundary condition

$$\phi_t(x) = 0, \quad x \in \mathbb{Z}^d \setminus D_N, t \geq 0,$$

and to clarify the relationship between the macroscopic motion and “Wulff shape” studied by [2]. The main result in this paper is that, under the scaling N^4 for time while N for space, the macroscopic motion corresponding to ϕ_t is described by the nonlinear partial differential equation with Dirichlet boundary condition

$$\left\{ \begin{aligned} \frac{\partial}{\partial t} h(t, \theta) &= -\Delta \operatorname{div} \left\{ (\nabla \sigma)(\nabla h(t, \theta)) \right\} \\ &\equiv \sum_{i=1}^d \sum_{j=1}^d \frac{\partial^2}{\partial \theta_j^2} \frac{\partial}{\partial \theta_i} \left\{ \frac{\partial \sigma}{\partial u_i} (\nabla h(t, \theta)) \right\}, \quad \theta \in D, t > 0 \\ h(t, \theta) &= 0, \quad \theta \in D^c, t > 0. \end{aligned} \right. \quad (1.3)$$

We note that the equation (1.3) is the gradient flow in an affine space in $(H^1(D))^*$ with respect to the energy functional

$$\Sigma_D(h) = \int_D \sigma(\nabla h(\theta)) d\theta$$

with the Dirichlet boundary condition $h|_{D^c} = 0$. Here, $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}$ is a function called “surface tension,” which gives the local energy of macroscopic interface with tilt $u \in \mathbb{R}^d$, see [5] for precise definition. The functional Σ_D is called “total surface tension,” which gives the total energy of interface h . We note that total surface tension Σ_D is the rate functional for the large deviation principle under the static situation, see [2]. Taking $\Gamma = \Gamma_N$ instead of D_N , the large scale hydrodynamic behavior has been studied by [6].

We should also mention the relationship between the equation (1.3) and the Wulff shape discussed in [2]. As an application of the large deviation principle, the macroscopic height variable h^N under the equilibrium state (Gibbs measure) conditioned on the total volume converges to the macroscopic interface so-called “Wulff shape,” the solution of the variational problem

$$\arg \inf \left\{ \Sigma_D(h); h \in H_0^1(D), \int_D h(\theta) d\theta = v \right\} \quad (1.4)$$

as $N \rightarrow \infty$, where v is the limit of the volume rescaled by N^{-d} . We emphasize that the solution $h(t)$ for (1.3) converges as $t \rightarrow \infty$, and the limit coincides with the solution of the variational problem (1.4). Indeed, the macroscopic motion described by (1.3) relaxes the total energy Σ_D and attains to the Wulff shape as the limit $t \rightarrow \infty$.

Before closing the introduction, let us give briefly the organization of this paper. In Section 2, we formulate our problem more precisely and state the main result. In Section 3, we study several properties of the macroscopic equation (1.3) and its spatial discretization. In Section 4, we show that a translation-invariant stationary measure for the dynamics ϕ_t on the infinite lattice need to be canonical Gibbs measure corresponding to the Hamiltonian H . Combining the above with the known result in [6], we have the characterization of the family of translation-invariant stationary measures. In Section 5, we derive the macroscopic equation (1.3) from the stochastic dynamics (1.2), after establishing several estimates for stochastic dynamics (2.1).

2. MODEL AND MAIN RESULTS

2.1. Model. Let D be a bounded, connected domain in \mathbb{R}^d with a Lipschitz boundary. For convenience, let D contain the origin of \mathbb{R}^d . In order to approximate profiles with the Neumann boundary condition by discrete ones, we will only consider D satisfying the following:

Assumption 2.1. Let $\tilde{D}_N = ND \cap \mathbb{Z}^d$. We assume that there exists a constant $C > 0$ independent of N such that

$$d_{\tilde{D}_N}(x, y) \leq C_2, \quad x, y \in \tilde{D}_N, |x - z| \leq 2, |y - z| \leq 2$$

for every $N \geq 1$ large enough and $z \in \mathbb{Z}^d \setminus \tilde{D}_N$, where $d_{\tilde{D}_N}$ is the ordinary graph distance on \tilde{D}_N .

Remark 2.1. Assumption 2.1 is satisfied if the domain D is convex.

Let us introduce the discretized microscopic domain corresponding to D . To keep notation simple, we shall consider $D_N \subset \mathbb{Z}^d$ defined by

$$D_N = \{x \in \tilde{D}_N; B(x/N, 5/N) \subset D\},$$

where $B(\alpha, l)$ stands for the hypercube in \mathbb{R}^d with center α and side length l , that is,

$$B(\alpha, l) = \prod_{i=1}^d [\alpha_i - l/2, \alpha_i + l/2].$$

On D_N we consider the dynamics governed by the following stochastic differential equations (SDEs)

$$d\phi_t(x) = -(-\Delta_{D_N})U(\phi_t)(x) dt + \sqrt{2}d\tilde{w}_t(x), \quad x \in D_N, \quad (2.1)$$

with the Dirichlet boundary condition

$$\phi_t(x) = 0, \quad x \in \mathbb{Z}^d \setminus D_N \quad (2.2)$$

and initial data ϕ_0 , where $U_x(\phi)$ in the drift term is defined by

$$U_x(\phi) := \frac{\partial H}{\partial \phi(x)}(\phi) \equiv \sum_{y \in \mathbb{Z}^d; |x-y|=1} V'(\phi(x) - \phi(y)) \quad (2.3)$$

for $\phi \in \mathbb{R}^{D_N}$ and $x \in D_N$. Here, Δ_{D_N} is the Laplace operator on D_N , that is, Δ_{D_N} is define by

$$\Delta_{D_N}\psi(x) = \sum_{y \in D_N; |x-y|=1} (\psi(y) - \psi(x))$$

for $\psi^N : \mathbb{R}^{D_N} \rightarrow \mathbb{R}$. The process $\tilde{w}_t = \{\tilde{w}_t(x); x \in D_N\}$ is a family of Gaussian processes with mean zero and covariance structure

$$E[\tilde{w}_t(x)\tilde{w}_s(x)] = (-\Delta_{D_N})(x, y)t \wedge s, \quad x, y \in D_N, t, s \geq 0.$$

Note that a stochastic processes \tilde{w}_t satisfying the above can be constructed by

$$\tilde{w}_t(x) = \sqrt{-\Delta_{D_N}} w_t(x),$$

where $\sqrt{-\Delta_{D_N}}$ is the square root of $-\Delta_{D_N}$ and $\{w_t(x); x \in D_N\}$ are a family of independent one dimensional Brownian motions. For convinience, we extend \tilde{w}_t to the process on \mathbb{Z}^d by putting $\tilde{w}_t(x) = 0$ when $x \in \mathbb{Z}^d \setminus D_N$.

Through out this paper, we always assume the following condition on V :

Assumption 2.2. The function $V : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the conditions as follows:

- (1) $V \in C^2(\mathbb{R})$.
- (2) V is symmetric, that is, $V(\eta) = V(-\eta)$ holds for all $\eta \in \mathbb{R}$.
- (3) There exist constants $c_+, c_- > 0$ such that

$$c_- \leq V''(\eta) \leq c_+, \quad \eta \in \mathbb{R}$$

holds.

We regard (2.1) as the model describing the motion of microscopic interfaces and introduce the macroscopic height variable h^N by scaling N^4 for time while N for space:

$$h^N(t, \theta) = \sum_{x \in \mathbb{Z}^d} N^{-1} \phi_{N^4 t}(x) 1_{B(x/N, 1/N)}(\theta), \quad \theta \in \mathbb{R}^d,$$

where $\phi_t = \{\phi_t(x); x \in \mathbb{Z}^d\}$ being the solution of (2.1) with (2.2). Note that the suitable scaling is not the diffusive one.

2.2. Main Result. The main result in this paper is the following:

Theorem 2.1. *We assume Assumptions 2.1 and 2.2. We furthermore assume that the sequence of initial data $\phi_0 = \phi_0^N$ for (2.1) satisfies*

$$\lim_{N \rightarrow \infty} E \|h^N(0) - h_0\|_{H^1(D)^*}^2 = 0 \quad (2.4)$$

with some $h_0 \in L^2(D)$, where $h^N(0)$ is the macroscopic height variable corresponding to ϕ_0^N . Then, for every $t > 0$, $h^N(t)$ converges as $N \rightarrow \infty$ to $h(t)$, which is the unique weak solution of the partial differential equation (1.3) with initial data h_0 . More precisely, for every $t > 0$,

$$\lim_{N \rightarrow \infty} E \|h^N(t) - h(t)\|_{H^1(D)^*}^2 = 0 \quad (2.5)$$

holds.

3. THE MACROSCOPIC EQUATION AND ITS DISCRETIZATION

In this section, we shall focus our attention on the limit equation (1.3) and its discretized version. The arguments in this section highly depend on the properties of the surface tension σ established in [2] and [5].

3.1. Sobolev space. As we will see, our computation is based on H^{-1} -norm. Before starting calculations, we introduce H^{-1} -norm and its discretization. Since the solution of (1.3) should satisfy

$$\int_D h(t, \theta) d\theta = \int_D h_0(\theta) d\theta, \quad t \geq 0,$$

which means that the actual state space for (1.3) is not linear. For convinience, let us mainly consider the time evolution on the tangential space. We denote the dual of the Sololev space $H^1(D)$ by $H^1(D)^*$. The equation (1.3) will be actually solved at

$$H = \{h \in H^1(D)^*; \langle h, 1 \rangle = 0\},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^1(D)^*$ and $H^1(D)$ such that

$$\langle f, g \rangle = (f, g)_{L^2(D)}$$

for every $f \in L^2(D)$, where $(\cdot, \cdot)_{L^2(D)}$ denotes the inner product in $L^2(D)$.

We shall next introduce the Green operator G for the Laplacian with zero Neumann boundary condition. For $h \in H$, we denote the unique solution $g \in H^1(D)$ of the elliptic equation

$$\begin{aligned} (\nabla g, \nabla J) &= \langle h, J \rangle, \quad J \in H^1(D), \\ (g, 1) &= 0 \end{aligned}$$

by Gf . Let us define the bilinear form $(\cdot, \cdot)_H$ on H by

$$(h_1, h_2)_H = (\nabla G h_1, \nabla G h_2), \quad h_1, h_2 \in H.$$

We then have that H is the Hilbert space with the inner product $(\cdot, \cdot)_H$, and that the norm associated to $(\cdot, \cdot)_H$ is equivalent to the ordinary $H^1(D)^*$ norm restricted to H .

Next, let us define the discrete version of the Hilbert space H define above, analogously. For the step function $f^N : \mathbb{R}^d \rightarrow \mathbb{R}$ with mesh size $1/N$, that is, f^N has the following representation

$$f^N(\theta) = \sum_{x \in \mathbb{Z}^d} N^{-1} \psi^N(x) 1_{B(x/N, 1/N)}(\theta)$$

with $\psi^N \in \mathbb{R}^{\mathbb{Z}^d}$, we define $\|f^N\|_{-1,N}$ by

$$\|f^N\|_{-1,N}^2 = N^{-d-4} \sum_{x \in D_N} (\psi^N(x) - \langle \psi^N \rangle) (-\Delta_{D_N})^{-1} (\psi^N(x) - \langle \psi^N \rangle) + N^{-2d-2} \langle \psi^N \rangle^2,$$

where

$$\langle \psi^N \rangle = \sum_{x \in D_N} \psi^N(x).$$

Note that the inverse of the Laplacian $(-\Delta_{D_N})^{-1}$ can be defined as the linear operator from

$$\mathcal{A}_N = \left\{ \phi \in \mathbb{R}^{D_N}; \sum_{x \in D_N} \phi(x) = 0 \right\}$$

to itself, by regarding Δ_{D_N} as 1:1 onto map from \mathcal{A}_N to \mathcal{A}_N . We can easily see that there exists a constant $C > 0$ independent of N such that

$$\|f^N\|_{H^1(D)^*} \leq C \|f^N\|_{-1,N}. \quad (3.1)$$

holds for every step function f^N with mesh size $1/N$ satisfying $f^N|_{D_N^c} \equiv 0$. We will frequently use (3.1) to establish a priori bounds and so on.

3.2. Precise formulation for macroscopic equation. We shall give the precise meaning of the solution of (1.3). Let us introduce a triple $V \subset H = H^* \subset V^*$ by H introduced in Section 3.1,

$$V = \left\{ h \in H_0^1(D); \int_D h(t, \theta) d\theta = 0 \right\}$$

and the dual space V^* . Note that the space V introduced above is closed subspace of $H_0^1(D)$ and therefore V is reflexive. We denote the duality relation between V^* and V by $v^* \langle \cdot, \cdot \rangle_V$, which satisfies

$$v^* \langle f, g \rangle_V = (f, g)_H$$

for $f \in H$ and $g \in V$. Let us consider the nonlinear fourth order differential operator $A_f : V \rightarrow V^*$ defined by

$$A_f(h) := -\Delta \operatorname{div} ((\nabla \sigma)(\nabla h + \nabla f)), \quad h \in V$$

for given $f \in H_0^1(D)$. We recall that the surface tension $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}$ is C^1 -class and it satisfies

$$c_- |u - v|^2 \leq (\nabla \sigma(u) - \nabla \sigma(v)) \cdot (u - v) \leq c_+ |u - v|^2, \quad u, v \in \mathbb{R}^d. \quad (3.2)$$

Definition 3.1. A function $h = h(t, \theta)$ is called the solution of (1.3) with initial data $h_0 \in H^1(D)^*$ if there exists a function $f \in H_0^1(D)$ with

$$\langle h_0, 1 \rangle = \langle f, 1 \rangle \quad (3.3)$$

such that the function $h_f := h - f$ satisfies the following conditions:

(1) $h_f : [0, T] \rightarrow V^*$ is absolutely continuous and

$$h_f \in C([0, T]; H) \cap L^2(0, T; V) \cap W^{1,2}([0, T], V^*).$$

(2) $h_f(0) = h_0 - f$.

(3) h_f satisfies

$$\frac{d}{dt}h_f = A_f(h_f(t))$$

in V^* for almost every $t \in [0, T]$.

Remark 3.1. The third condition is equivalent to

$$h_f(t) = h_f(0) + \int_0^t A_f(h_f(s)) ds$$

in V^* for almost every $t \in [0, T]$. Roughly saying, the above is equivalent to

$$h(t) = h(0) + \int_0^t A(h(s)) ds,$$

where A is the nonlinear fourth order differential operator defined by

$$A(h) := -\Delta \operatorname{div}((\nabla \sigma)(\nabla h)).$$

The first aim in this section is to see the existence and uniqueness for our equation.

Theorem 3.2. *For every $h_0 \in H^1(D)^*$, there exists a unique solution of (1.3).*

In order to apply a general theory of nonlinear partial differential equations, we prepare the following lemma:

Lemma 3.3. *For every $f \in H_0^1(D)$, the nonlinear operator $A_f : V \rightarrow V^*$ satisfies the following:*

(1) A is monotone, that is,

$$_{V^*} \langle A_f(h_1) - A_f(h_2), h_1 - h_2 \rangle_V \leq 0, \quad h_1, h_2 \in V.$$

holds.

(2) A is demicontinuous, that is, $A_f(h)$ is continuous under the weak topology of V^* .

(3) There exist constants $C_1, C_2, C_3 > 0$ such that

$$_{V^*} \langle A_f(h), h \rangle_V \leq -C_1 \|h\|_V^2 + C_2, \quad h \in V, \quad (3.4)$$

$$\|A_f(h)\|_{V^*} \leq C_3(\|h\|_V + 1), \quad h \in V \quad (3.5)$$

holds.

Proof. We at first note

$$_{V^*} \langle A_f(h), g \rangle_V = -((\nabla \sigma)(\nabla h + \nabla f), \nabla g)_{L^2(D)} \quad (3.6)$$

for $h, g \in V$, and

$$_{V^*} \langle A_f(h_1) - A_f(h_2), g \rangle_V = -((\nabla \sigma)(\nabla h_1 + \nabla f) - (\nabla \sigma)(\nabla h_2 + \nabla f), \nabla g)_{L^2(D)} \quad (3.7)$$

for $h_1, h_2, g \in V$. It is easy to see (1), by applying (3.7) with $g = h_1 - h_2$ and using the convexity of σ , see (3.2). We can also obtain (2), since we have

$$|_{V^*} \langle A_f(h_1) - A_f(h_2), g \rangle_V| \leq c_+ \|\nabla h_1 - \nabla h_2\|_{L^2(D)} \|\nabla g\|_{L^2(D)}, \quad h_1, h_2, g \in V \quad (3.8)$$

from (3.2) and (3.7) again. Moreover, we can obtain (3.5) of (3) also, because the relationship (3.6) implies

$$\|A_f(h)\|_{V^*} \leq \|(\nabla\sigma)(\nabla h + \nabla f)\|_{L^2(D)} \leq c_+ \|h\|_V + c_+ \|\nabla f\|_{L^2(D)}$$

for $h \in V$.

As the final step, we shall show (3.4) of (3). Using (3.6), we obtain

$$\begin{aligned} {}_{V^*}\langle A_f(h), h \rangle_V &= -((\nabla\sigma)(\nabla h + \nabla f), \nabla h)_{L^2(D)} \\ &= -((\nabla\sigma)(\nabla h + \nabla f), \nabla h + \nabla f)_{L^2(D)} + ((\nabla\sigma)(\nabla h + \nabla f), \nabla f)_{L^2(D)} \\ &\leq -c_- \|\nabla h + \nabla f\|_{L^2(D)}^2 + c_+ \|\nabla h + \nabla f\|_{L^2(D)} \|\nabla f\|_{L^2(D)}. \end{aligned}$$

Here, noting

$$\|\nabla h + \nabla f\|_{L^2(D)}^2 \geq \frac{1}{2} \|\nabla h\|_{L^2(D)}^2 - \|\nabla f\|_{L^2(D)}^2,$$

and

$$c_+ \|\nabla h + \nabla f\|_{L^2(D)} \|\nabla f\|_{L^2(D)} \leq \frac{1}{4} c_- \|\nabla h\|_{L^2(D)}^2 + \left(\frac{4c_+^2}{c_-} + c_+ \right) \|\nabla f\|_{L^2(D)}^2$$

we obtain

$${}_{V^*}\langle A_f(h), h \rangle_V \leq -\frac{1}{4} c_- \|\nabla h\|_{L^2(D)}^2 + \left(\frac{4c_+^2}{c_-} + c_+ + c_- \right) \|\nabla f\|_{L^2(D)}^2.$$

Applying the Poincaré inequality for $h \in V \subset H_0^1(D)$, we conclude the desired bound (3.4). \square

Proof of Theorem 3.2. Using Theorem 4.10 of [1] and Lemma 3.3, for every initial data $h_0 \in H^1(D)^*$ and every auxiliary function $f \in H_0^1(D)$ satisfying (3.3), we obtain the existence and uniqueness of h_f satisfying conditions (1)-(3) in Definition 3.1. Especially, this shows the existence of the solution of (1.3) in the sense of Definition 3.1.

Let us show the uniqueness of the solution of (1.3) in the sense of Definition 3.1. Take a two solution $h^{(1)}$ and $h^{(2)}$ with a common initial data h_0 , and let f_1 and f_2 be auxiliary functions associated to $h^{(1)}$ and $h^{(2)}$, respectively. Noting

$$A_{f_1}(h) = A_{f_2}(h - f_2 + f_1), \quad h \in V,$$

$h^{(2)} - f_1 = h_{f_2}^{(2)} + f_2 - f_1$ satisfies conditions (1)-(3) in Definition 3.1 with f_1 . The uniqueness of h_f for given f implies $h^{(2)} - f_1 = h^{(1)} - f_1$, which shows $h^{(1)} = h^{(2)}$. \square

3.3. Regularization for the macroscopic equation. Let us introduce the regularization of σ and the corresponding partial differential equation, which plays key role in the proof of Theorem 2.1. Note that such regularization is not needed once one can solve C^2 -regularity of σ .

Let $\rho \in C_0^\infty(\mathbb{R}^d)$ be non-negative, symmetric, $\text{supp } \rho \subset \{u \in \mathbb{R}^d; |u| < 1\}$ and $\int_{\mathbb{R}^d} \rho(x) dx = 1$. For $0 \leq \delta \leq 1$, we define ρ_δ by

$$\rho_\delta = \delta^{-d} \rho(\delta^{-1}u), \quad u \in \mathbb{R}^d$$

and the regularized surface tension σ^δ by the mollification of σ :

$$\sigma^\delta(u) = \sigma * \rho_\delta(u), \quad u \in \mathbb{R}^d.$$

Note that the regularized surface tension σ^δ again satisfies the bound (3.2), that is,

$$c_-|u - v|^2 \leq (\nabla\sigma^\delta(u) - \nabla\sigma^\delta(v)) \cdot (u - v) \leq c_+|u - v|^2, \quad u, v \in \mathbb{R}^d \quad (3.9)$$

holds. Moreover, since $\nabla\sigma$ is continuous, σ^δ approximate σ in the following sense:

$$|\nabla\sigma^\delta(u) - \nabla\sigma(u)| \leq c_+\delta, \quad u \in \mathbb{R}^d. \quad (3.10)$$

Using σ^δ defined above, let us consider the nonlinear fourth order differential equation

$$\begin{cases} \frac{\partial}{\partial t} h^\delta(t, \theta) = -\Delta \operatorname{div} \left\{ (\nabla\sigma^\delta)(\nabla h(t, \theta)) \right\}, & \theta \in D, t > 0 \\ h^\delta(t, \theta) = 0, & \theta \in D^c, t > 0. \end{cases} \quad (3.11)$$

The equation (3.11) can be formulated by similar way to Definition 3.1. Since we have (3.9), Theorem 3.2 can be applied to (3.11) also. We therefore obtain that the equation (3.11) has a unique solution. Furthermore, we get the following proposition, which implies that the solution of (1.3) can be approximated by the solution of (3.11).

Proposition 3.4. *Let h and h^δ be the solutions of (1.3) and (3.11), respectively. If the initial datum are common, we then have*

$$\lim_{\delta \rightarrow 0} \|h(t) - h^\delta(t)\|_{H^1(D)^*} = 0.$$

Proof. Let us fix an auxiliary function $f \in H_0^1(D)$. By Definition 3.1, we obtain

$$\begin{aligned} \|h^\delta(t) - f\|_H^2 &= \|h^\delta(0) - f\|_H^2 - 2 \int_0^t (\nabla\sigma^\delta(\nabla h^\delta(s)) - \nabla\sigma^\delta(\nabla f), \nabla h^\delta(s) - \nabla f)_{L^2(D)} ds \\ &\quad - 2 \int_0^t (\nabla\sigma^\delta(\nabla f), \nabla h^\delta(s) - \nabla f)_{L^2(D)} ds \\ &\leq \|h^\delta(0) - f\|_H^2 - 2c_- \int_0^t \|\nabla h^\delta(s) - \nabla f\|_{L^2(D)}^2 ds \\ &\quad + 2 \int_0^t \|\nabla\sigma^\delta(\nabla f)\|_{L^2(D)} \|\nabla h^\delta(s) - \nabla f\|_{L^2(D)} ds \\ &\leq \|h^\delta(0) - f\|_H^2 - c_- \int_0^t \|\nabla h^\delta(s) - \nabla f\|_{L^2(D)}^2 ds \\ &\quad + c_-^{-1} \int_0^t \|\nabla\sigma^\delta(\nabla f)\|_{L^2(D)}^2 ds \end{aligned}$$

from (3.6) and (3.9). This shows

$$\sup_{\delta > 0} \int_0^t \|h^\delta(s)\|_{H^1(D)}^2 ds < \infty. \quad (3.12)$$

By similar calculation to the above, we obtain

$$\begin{aligned}
& \|h(t) - h^\delta(t)\|_H^2 \\
&= -2 \int_0^t (\nabla \sigma(\nabla h(s) + \nabla f) - \nabla \sigma(\nabla h^\delta(s) + \nabla f), \nabla h(s) - \nabla h^\delta(s))_{L^2(D)} ds \\
&\quad - 2 \int_0^t (\nabla \sigma(\nabla h^\delta(s) + \nabla f) - \nabla \sigma^\delta(\nabla h^\delta(s) + \nabla f), \nabla h(s) - \nabla h^\delta(s))_{L^2(D)} ds \\
&\leq 2 \int_0^t c_+ \delta \int_D |\nabla h(s, \theta) - \nabla h^\delta(s, \theta)| d\theta ds.
\end{aligned}$$

Combining above with (3.12), we get the conclusion. \square

We can show the following proposition by similar argument to the proof of Proposition 3.4.

Proposition 3.5. *Let both of h and \hat{h} be the solutions of (1.3). We then have*

$$\|h(t) - \hat{h}(t)\|_H^2 \leq \|h(0) - \hat{h}(0)\|_H^2, \quad t \geq 0.$$

3.4. The discretization for the macroscopic equation. In order to introduce the discretized equation corresponding to the regularized macroscopic equation (3.11), let us introduce several notations. We define the finite difference operators by

$$\begin{aligned}
\nabla_i^N f(\theta) &= N(f(\theta + e_i/N) - f(\theta)), \\
\nabla_i^{N,*} f(\theta) &= -N(f(\theta) - f(\theta - e_i/N)), \\
\nabla^N f(\theta) &= (\nabla_1^N f(\theta), \dots, \nabla_d^N f(\theta)), \\
\operatorname{div}_N g(\theta) &= - \sum_{i=1}^d \nabla_i^{N,*} g_i(\theta)
\end{aligned}$$

for $f : \mathbb{R}^d \rightarrow \mathbb{R}$, $g = (g_i)_{1 \leq i \leq d} : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $1 \leq i \leq d$ and $\theta \in \mathbb{R}^d$, where $e_i \in \mathbb{Z}^d$ is the i -th unit vector given by $(e_i)_j = \delta_{ij}$ for $1 \leq i \leq d$. We also define the discretized Laplacian with the Neumann boundary condition by

$$\Delta_N f(\theta) = N \sum_{i=1}^d \left(\nabla_i^N f(\theta) 1_{\theta \in \tilde{D}_N} 1_{\theta + e_i/N \in \tilde{D}_N} - \nabla_i^N f(\theta - e_i/N) 1_{\theta \in \tilde{D}_N} 1_{\theta - e_i/N \in \tilde{D}_N} \right)$$

for $f : \mathbb{R}^d \rightarrow \mathbb{R}$, where the domain \tilde{D}_N is defined by

$$\tilde{D}_N = \bigcup_{x \in D_N} B(x/N, 1/N).$$

Note that indicator functions appearing above are corresponding to the range of x 's where the sum (2.3) is taken. With these notations the discretized PDE for (3.11) reads a system

of ordinary differential equations

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} \bar{h}^{N,\delta}(t, \theta) = -\Delta_N k^{N,\delta} \\ k^{N,\delta} = A^{N,\delta}(\bar{h}^{N,\delta}(t))(\theta) \\ \quad \quad \quad := \operatorname{div}_N \{ (\nabla \sigma^\delta)(\nabla^N \bar{h}^{N,\delta}(t)) \}(\theta), \quad \theta \in \tilde{D}_N, \\ \bar{h}^{N,\delta}(t, \theta) = 0, \quad \theta \notin \tilde{D}_N. \end{array} \right. \quad (3.13)$$

We recall that $\sigma^\delta \in C^\infty(\mathbb{R}^d)$ and it satisfies (3.9). The equation (3.13) will be solved with the initial data given by

$$\bar{h}_0^{N,\delta}(\theta) = N^d \int_{B(x/N, 1/N)} h_0(\theta) d\theta, \quad \theta \in B(x/N, 1/N), x \in \mathbb{Z}^d. \quad (3.14)$$

where $h_0 \in L^2(\mathbb{R}^d)$. Since the initial data h_0^N is a step function, the solution $\bar{h}^N(t, \theta)$ is also a step function, that is,

$$\bar{h}^{N,\delta}(t, \theta) = \bar{h}^{N,\delta}(t, x/N), \quad \theta \in B(x/N, 1/N), x \in \mathbb{Z}^d.$$

Though the equation (3.13) relies on the value of $k^{N,\delta}$ in D_N only, let us extend $k^{N,\delta}$ to the domain large enough for convinience. We define $\partial_i D_N$ by

$$\partial_i D_N = \{x \in \mathbb{Z}^d \setminus D_N; \operatorname{dist}_{\mathbb{Z}^d}(x, D_N) = i\},$$

where $\operatorname{dist}_{\mathbb{Z}^d}$ is the graph distance on \mathbb{Z}^d . We define the value of u_N on $\partial_k D_N$ recursively, by

$$k^{N,\delta}(x) = (\#\{y \in D_N; |x - y| = 1\})^{-1} \sum_{y \in D_N; |x-y|=1} k^{N,\delta}(y), \quad x \in \partial_1 D_N,$$

and

$$k^{N,\delta}(x) = (\#\{y \in \partial_{i-1} D_N; |x - y| = 1\})^{-1} \sum_{y \in \partial_{i-1} D_N; |x-y|=1} k^{N,\delta}(y), \quad x \in \partial_i D_N.$$

3.5. A priori bound for the discretized equation. In this section, we establish a priori bound of the solution $\bar{h}^{N,\delta}$ of (3.13), which is uniform in the mesh size N . To do so, we introduce an auxiliary function similarly to Definition 3.1. Let us take the function $g \in C_0^\infty(D)$ which satisfies the following:

- (1) $g(x) \geq 0$ holds for every $x \in D$.
- (2) $\int_D g(\theta) d\theta = 1$.

We can then take $N_0 \geq 1$ enough large such that $\operatorname{supp} g \subset \tilde{D}_N$ for every $N \geq N_0$. Let us establish a priori bound for \bar{h}^N with $N \geq N_0$.

Using g introduced above, we define ζ^N by

$$\zeta^N(x) = N^{d+1} \int_{B(x/N, 1/N)} g(\theta) d\theta, \quad x \in \mathbb{Z}^d$$

and g^N by

$$g^N(\theta) = \sum_{x \in \mathbb{Z}^d} N^{-1} \psi^N(x) 1_{B(x/N, 1/N)}(\theta), \quad \theta \in \mathbb{R}^d.$$

We then have the following bound:

$$\sup_{N \geq 1} \sup_{1 \leq i, j \leq d} \sup_{x \in \overline{D_N}} \{ |g^N(x/N)| + |\nabla_i^N g^N(x/N)| + |\nabla_i^N \nabla_j^N g^N(x/N)| \} \leq c_g, \quad (3.15)$$

where c_g is the constant defined by

$$c_g = \sup_{1 \leq i, j \leq d} \left\{ \|g\|_\infty + \left\| \frac{\partial g}{\partial \theta_i} \right\|_\infty + \left\| \frac{\partial^2 g}{\partial \theta_i \partial \theta_j} \right\|_\infty + 1 \right\}.$$

Using ζ^N and g^N introduced above, we define ψ^N and f^N by

$$\psi^N(x) = v \zeta^N(x), \quad x \in \mathbb{Z}^d \quad (3.16)$$

and

$$f^N(x) = v g^N(\theta), \quad \theta \in \mathbb{R}^d \quad (3.17)$$

respectively, where v is the “volume” of h_0 , that is,

$$v = \int_D \bar{h}_0^{N,\delta}(\theta) d\theta = \langle h_0, 1 \rangle$$

for the initial datum \bar{h}_0^N for (3.13). Note that the right hand side of the above does not depend on the choice of δ . We then have that sequences $\{\psi^N\}$ and $\{f^N\}$ satisfy following properties:

- (1) For every N and $x \in \mathbb{Z}^d \setminus D_N$, $\psi^N(x) = f^N(x/N) = 0$ holds.
- (2) For every N , the relationship

$$\sum_{x \in D_N} \psi^N(x) = N \int_D f^N(\theta) d\theta$$

holds.

- (3) The following bounds hold:

$$\sup_{N \geq 1} \sup_{1 \leq i, j \leq d} \sup_{x \in \overline{D_N}} \{ |f^N(x/N)| + |\nabla_i^N f^N(x/N)| + |\nabla_i^N \nabla_j^N f^N(x/N)| \} \leq c_g |v|. \quad (3.18)$$

Proposition 3.6. *There exist constants C_1, C_2 independent of N, δ such that*

$$\|\bar{h}^{N,\delta}(t)\|_{-1,N}^2 + N^{-d} \int_0^t \|\nabla^N \bar{h}^{N,\delta}(s)\|_{L^2(D)}^2 ds \leq C_1 \|\bar{h}^{N,\delta}(0)\|_{-1,N}^2 + C_2(1+t)$$

holds for every $t \geq 0$.

Proof. Differentiating $\|\bar{h}^{N,\delta}(t) - f^N\|_{-1,N}^2$ in t , we obtain

$$\begin{aligned} \frac{d}{dt} \|\bar{h}^{N,\delta}(t) - f^N\|_{-1,N}^2 &= -2N^{-d} \sum_{x \in \overline{D_N}} \nabla^N \bar{h}^{N,\delta}(t, x/N) \cdot \nabla \sigma^\delta(\nabla^N \bar{h}^{N,\delta}(t, x/N)) \\ &\quad + 2N^{-d} \sum_{x \in \overline{D_N}} \nabla^N f^N(x/N) \cdot \nabla \sigma^\delta(\nabla^N \bar{h}^{N,\delta}(t, x/N)) \\ &\leq -2c_- N^{-d} \sum_{x \in \overline{D_N}} |\nabla^N \bar{h}^{N,\delta}(t, x/N)|^2 \\ &\quad + 2N^{-d} \sum_{x \in \overline{D_N}} \nabla^N f^N(x/N) \cdot \nabla \sigma^\delta(\nabla^N \bar{h}^{N,\delta}(t, x/N)), \end{aligned} \quad (3.19)$$

where $\overline{D_N}$ is defined by

$$\overline{D_N} = \{x \in \mathbb{Z}^d; \text{ there exists } y \in D_N \text{ such that } |x - y| \leq 1\}.$$

Here, we have used the summation-by-parts formula

$$\sum_{x \in D_N} \alpha(x/N) \operatorname{div}_N \beta(x/N) = - \sum_{x \in \overline{D_N}} \nabla^N \alpha(x/N) \cdot \beta(x/N), \quad (3.20)$$

where $\alpha : \mathbb{R}^d \rightarrow \mathbb{R}$ and $\beta = (\beta_i)_{i=1}^d : \mathbb{R}^d \rightarrow \mathbb{R}^d$ are arbitrary functions such that α and β_i ($1 \leq i \leq d$) are step functions with mesh size $1/N$ and $\alpha(x/N) = 0$ for every $x \in \mathbb{Z}^d \setminus \overline{D_N}$.

The second term in the right hand side can be estimated in the following way:

$$\begin{aligned} & \left| N^{-d} \sum_{x \in \overline{D_N}} \nabla^N f^N(x/N) \cdot \nabla \sigma^\delta(\nabla^N \bar{h}^{N,\delta}(t, x/N)) \right| \\ & \leq \frac{1}{2} c_- N^d \sum_{x \in \overline{D_N}} |\nabla^N \bar{h}^{N,\delta}(t, x/N)|^2 + C N^{-d} |v| |\overline{D_N}|, \end{aligned}$$

with a constant $C > 0$ independent of N . We have used the properties of f^N stated at the beginning of this subsection. Plugging the above into (3.19) and integrating in t , we obtain

$$\begin{aligned} \|\bar{h}^{N,\delta}(T) - f^N\|_{-1,N}^2 & \leq \|\bar{h}^{N,\delta}(0) - f^N\|_{-1,N}^2 \\ & \quad - \frac{1}{2} c_- \int_0^T \|\nabla^N \bar{h}^{N,\delta}(t)\|_{L^2(D)}^2 dt + C N^{-d} |v| |\overline{D_N}| T, \end{aligned}$$

which implies the desired estimate, since $\|f^N\|_{-1,N}$ is bounded uniformly in N . \square

We can improve the bound for $\nabla^N \bar{h}^{N,\delta}$ if the initial datum is smooth enough.

Proposition 3.7. *We assume that*

$$h_0 \in C_0^\infty(D) \quad (3.21)$$

We then have the following uniform bound:

$$\sup_N \sup_{0 \leq t \leq T} \|\nabla^N \bar{h}^{N,\delta}(t)\|_{L^2(D)}^2 < \infty$$

for every $T > 0$.

Proof. Differentiating $\sum_{x \in \overline{D_N}} \sigma^\delta(\nabla^N \bar{h}^{N,\delta}(t, x/N))$ in t , we have

$$\begin{aligned} \frac{\partial}{\partial t} \sum_{x \in \overline{D_N}} \sigma^\delta(\nabla^N \bar{h}^{N,\delta}(t, x/N)) & = \sum_{x \in \overline{D_N}} \nabla \sigma^\delta(\nabla^N \bar{h}^{N,\delta}(t, x/N)) \nabla^N \frac{\partial}{\partial t} \bar{h}^{N,\delta}(t, x/N) \\ & = \sum_{x \in D_N} \operatorname{div}_N \nabla \sigma^\delta(\nabla^N \bar{h}^{N,\delta}(t, x/N)) \frac{\partial}{\partial t} \bar{h}^{N,\delta}(t, x/N) \\ & = \sum_{x \in D_N} k^{N,\delta}(t, x/N) \Delta_N k^{N,\delta}(t, x/N). \end{aligned} \quad (3.22)$$

Here, we have used (3.8) in [7], since

$$\frac{\partial \bar{h}^N}{\partial t}(t, x/N) = 0, \quad x \in D_N^c$$

holds. Since $-\Delta_N$ is non-negative definite, we obtain that the right hand side is non-positive. Dropping the right hand side and integrating in t , we have

$$\sum_{x \in \overline{D_N}} \sigma^\delta(\nabla^N \bar{h}^{N,\delta}(t, x/N)) \leq \sum_{x \in \overline{D_N}} \sigma^\delta(\nabla^N \bar{h}^{N,\delta}(0, x/N)),$$

which indicate the conclusion, since the function σ satisfies (3.2). \square

Let us establish the bound for k^N in (3.13), in order to apply the argument in Section 3.3 of [7].

Proposition 3.8. *We assume (3.21). We then have the following bound:*

$$\sup_N \left\{ \sup_{0 \leq t \leq T} \left\| \frac{\partial \bar{h}^{N,\delta}}{\partial t} \right\|_{-1,N}^2 + \int_0^T \left\| \nabla^N \frac{\partial \bar{h}^{N,\delta}}{\partial t} \right\|_{L^2(D)}^2 dt \right\} < \infty.$$

Proof. Noting

$$\frac{\partial^2 \bar{h}^{N,\delta}}{\partial t^2}(t, x/N) = \sum_{i,j=1}^d \Delta_N \nabla_i^{N,*} \left\{ \frac{\partial^2 \sigma^\delta}{\partial u_i \partial u_j}(\nabla^N \bar{h}^{N,\delta}) \nabla_j^N \frac{\partial \bar{h}^{N,\delta}(t)}{\partial t} \right\}$$

and

$$\sum_{x \in D_N} \frac{\partial \bar{h}^{N,\delta}}{\partial t}(t, x/N) = 0,$$

we have

$$\begin{aligned} \frac{d}{dt} \left\| \frac{\partial \bar{h}^{N,\delta}}{\partial t} \right\|_{-1,N}^2 &= -2N^{-d} \sum_{i,j=1}^d \sum_{x \in \overline{D_N}} \nabla_i^N \frac{\partial \bar{h}^{N,\delta}}{\partial t}(t, x/N) \frac{\partial^2 \sigma}{\partial u_i \partial u_j}(\nabla^N \bar{h}^{N,\delta}) \nabla_j^N \frac{\partial \bar{h}^{N,\delta}(t)}{\partial t} \\ &\leq -2c_- N^{-d} \sum_{i=1}^d \sum_{x \in \overline{D_N}} \left(\nabla_i^N \frac{\partial \bar{h}^{N,\delta}}{\partial t}(t, x/N) \right)^2, \end{aligned}$$

by performing the summation-by-parts several times. Integrating the both sides in t , we have

$$\left\| \frac{\partial \bar{h}^{N,\delta}}{\partial t} \right\|_{-1,N}^2 + 2c_- N^{-d} \int_0^t \sum_{i=1}^d \sum_{x \in \overline{D_N}} \left(\nabla_i^N \frac{\partial \bar{h}^{N,\delta}}{\partial s}(s, x/N) \right)^2 ds \leq \left\| \frac{\partial \bar{h}^N}{\partial t}(0) \right\|_{-1,N}^2$$

which implies the conclusion. \square

Remark 3.2. By the definition of $k^{N,\delta}$, we have

$$\left\| \frac{\partial \bar{h}^{N,\delta}}{\partial t} \right\|_{-1,N}^2 = N^{-d} \sum_{x \in D_N} k^{N,\delta}(t, x/N) (-\Delta_N k^{N,\delta})(t, x/N).$$

We therefore obtain

$$\sup_N \sup_{0 \leq t \leq T} N^{-d} \sum_{x \in D_N} k^{N,\delta}(t, x/N) (-\Delta_N k^{N,\delta})(t, x/N) < \infty.$$

by Proposition 3.8.

Remark 3.3. We need the smoothness of σ^δ in order to obtain the uniform bound in N for

$$N^{-d} \sum_{x \in D_N} k^{N,\delta}(0, x/N) (-\Delta_N k^{N,\delta})(0, x/N),$$

even if h_0 is smooth, for example, $h_0 \in C_0^\infty(D)$. We have C^1 -regularity of σ and the Lipschitz continuity of $\nabla \sigma$, but such regularity is less than that we need. It is the reason why we consider the equation (3.13) with smooth σ^δ instead of the original surface tension σ .

As a direct consequence of Proposition 3.8, we have the following result.

Corollary 3.9. *We assume (3.21). There exists a constant $C > 0$ independent of N such that*

$$\sup_N \left\| \bar{h}^{N,\delta}(t_1) - \bar{h}^{N,\delta}(t_2) \right\|_{-1,N}^2 \leq C |t_1 - t_2|$$

holds for every $0 \leq t_1, t_2 \leq T$.

3.6. Uniform L^2 -bound for $k^{N,\delta}$. In this subsection, we shall establish the uniform L^2 -bound for $k^{N,\delta}$. Our goal in this section is the following:

Proposition 3.10. *Under the assumption (3.21), we obtain*

$$\sup_N \sup_{0 \leq t \leq T} |\langle k^{N,\delta}(t) \rangle| < \infty \quad (3.23)$$

for every $T > 0$, where

$$\langle k^{N,\delta}(t) \rangle = N^{-d} \sum_{x \in D_N} k^{N,\delta}(t, x/N).$$

In order to show the above, we shall reduce our problem to that for solutions of elliptic equations whose main term is linear. We at first note that the gradient of the surface tension is expressed by the expected value of $V'(\eta(b))$:

$$\nabla_i \sigma(u) = E^{\mu_u} [V'(\eta(e_i, 0))], \quad 1 \leq i \leq d,$$

where e_i is the i -th unit vector in \mathbb{R}^d . For simplicity, we denote the directed bond $(e_i, 0)$ simply by e_i again. We then have the following decomposition for $\nabla \sigma$:

$$\nabla \sigma(u) = A(u)u + a(u), \quad (3.24)$$

where $A(u) = (A_{ij}(u))$ is the $d \times d$ diagonal matrix defined by

$$A_{ij}(u) = E^{\mu_u} \left[\int_0^1 V''(\eta(e_i) - \lambda u_i) d\lambda \right] \delta_{ij}, \quad 1 \leq i, j \leq d$$

and $a(u) = (a_i(u)) \in \mathbb{R}^d$ is defined by

$$a_i(u) = E^{\mu_u} [V'(\eta(e_i) - u_i)], \quad 1 \leq i \leq d.$$

We remark that the matrix $A(u)$ satisfies

$$c_- \mathbb{I} \leq A(u) \leq c_+ \mathbb{I} \quad (3.25)$$

uniformly in u , where \mathbb{I} is the $d \times d$ identity matrix. Furthermore, we also remark that the vector a satisfies

$$\sup_{1 \leq i \leq d} \sup_{u \in \mathbb{R}^d} |a_i(u)| \leq C_a$$

with some constant $C_a > 0$, which is a simple consequence of Brascamp-Lieb inequality, see [2] and [5]. By (3.24) and the definition of σ^δ , we also obtain the decomposition for $\nabla \sigma^\delta$ as follows:

$$\nabla \sigma^\delta(u) = A^\delta(u)u + a^\delta(u),$$

where

$$A_{ij}^\delta(u) = A_{ij} * \rho_\delta(u), \quad 1 \leq i, j \leq d$$

and

$$a_i^\delta(u) = a_{ij} * \rho_\delta(u) + \int_{\mathbb{R}^d} A_{ii}(u-v) v_i \rho_\delta(v) dv, \quad 1 \leq i \leq d.$$

We note that A^δ is diagonal and satisfies (3.25) again, and that a^δ satisfies

$$\sup_{\delta > 0} \sup_{1 \leq i \leq d} \sup_{u \in \mathbb{R}^d} |a_i^\delta(u)| < C'_a$$

with some constant $C'_a > 0$. Putting

$$\begin{aligned} A^{N,\delta}(t, \theta) &= A^\delta(\nabla^N \bar{h}^{N,\delta}(t, \theta)), \\ a^{N,\delta}(t, \theta) &= a^\delta(\nabla^N \bar{h}^{N,\delta}(t, \theta)), \end{aligned}$$

we see that $\bar{h}^{N,\delta}(t)$ satisfies

$$k^{N,\delta}(t) = \operatorname{div}_N(A^{N,\delta}(t) \nabla^N \bar{h}^{N,\delta}(t)) + \operatorname{div}_N a^{N,\delta}(t), \quad t \geq 0. \quad (3.26)$$

Let us regard (3.26) as the elliptic equation for given $A^{N,\delta}(t)$, $a^{N,\delta}(t)$ and $k^{N,\delta}$, which has a unique solution. Since the main term of (3.26) is linear, we can write $\bar{h}^{N,\delta}$ as the sum of profiles. For $t > 0$, let $\bar{h}_1^{N,\delta}(t)$ be the unique solution of

$$\tilde{k}^{N,\delta}(t) = \operatorname{div}_N(A^{N,\delta}(t) \nabla^N \bar{h}_1^{N,\delta}(t)) + \operatorname{div}_N a^{N,\delta}(t) \quad (3.27)$$

with the Dirichlet boundary condition

$$\bar{h}_1^{N,\delta}(t, x/N) = 0, \quad x \in \mathbb{Z}^d \setminus D_N,$$

where $\tilde{u}^{N,\delta}(t)$ is defined by

$$\tilde{k}^{N,\delta}(t) := k^{N,\delta}(t) - \langle k^{N,\delta}(t) \rangle.$$

Furthermore, we let $\bar{h}_2^{N,\delta}(t)$ be the unique solution of

$$\langle k^{N,\delta}(t) \rangle = \operatorname{div}_N(A^{N,\delta}(t) \nabla^N \bar{h}_2^{N,\delta}(t)) \quad (3.28)$$

with the Dirichlet boundary condition similarly to \bar{h}_1^N . Note that the original $\bar{h}^N(t)$ can be recovered by

$$\bar{h}^N(t) = \bar{h}_1^N(t) + \bar{h}_2^N(t). \quad (3.29)$$

We shall at first show the bound for \bar{h}_1^N .

Proposition 3.11. *There exist constants $C_1, C_2 > 0$ independent of N such that*

$$\|\nabla^N \bar{h}_1^N(t)\|_{L^2(D)}^2 + \|\bar{h}_1^N(t)\|_{L^2(D)}^2 \leq C_1 \|\tilde{k}^{N,\delta}(t)\|_{L^2(D)}^2 + C_2$$

holds for every $t \geq 0$.

Proof. Multiplying the both side of (3.27) by $\bar{h}_1^N(t)$ and taking the sum over D_N , we have

$$\begin{aligned} \sum_{x \in D_N} \tilde{k}^{N,\delta}(t, x/N) \bar{h}_1^N(t, x/N) &= \sum_{x \in D_N} \bar{h}_1^N(t, x/N) \operatorname{div}_N(A^N(t) \nabla^N \bar{h}_1^N(t))(x/N) \\ &\quad + \sum_{x \in D_N} \bar{h}_1^N(t, x/N) \operatorname{div}_N a^N(t)(x/N). \end{aligned}$$

Dividing the both side by N^d and performing the summation-by-parts, we obtain

$$\begin{aligned} N^{-d} \sum_{x \in D_N} \tilde{k}^{N,\delta}(t, x/N) \bar{h}_1^N(t, x/N) &= N^{-d} \sum_{x \in \overline{D_N}} \nabla^N \bar{h}_1^N(t, x/N) \cdot A^N(t) \nabla^N \bar{h}_1^N(t, x/N) \\ &\quad + N^{-d} \sum_{x \in \overline{D_N}} \nabla \bar{h}_1^N(t, x/N) a^N(t, x/N) \\ &\leq -\frac{1}{2} c_- \sum_{x \in \overline{D_N}} |\nabla^N \bar{h}_1^N(t, x/N)|^2 + \frac{8C_a}{c_-} N^d |D_N|. \end{aligned}$$

We have used (3.25) and $\bar{h}_1^N(t, x/N) = 0$ for $x \in \mathbb{Z}^d \setminus D_N$. Using the Poincaré inequality

$$\|\bar{h}_1^N(t)\|_{L^2}^2 \leq C \|\nabla^N \bar{h}_1^N(t)\|_{L^2}^2$$

for a constant $C > 0$ independent in N , we have

$$\begin{aligned} \frac{1}{2} c_- \|\nabla^N \bar{h}_1^N(t)\|_{L^2}^2 &\leq 2\gamma \|\tilde{k}^{N,\delta}(t)\|_{L^2}^2 + 2\gamma^{-1} \|\bar{h}_1^N(t)\|_{L^2}^2 + \frac{8C_a}{c_-} N^d |D_N| \\ &\leq 2\gamma \|\tilde{k}^{N,\delta}(t)\|_{L^2}^2 + 2\gamma^{-1} C \|\nabla \bar{h}_1^N(t)\|_{L^2}^2 + \frac{8C_a}{c_-} N^d |D_N| \end{aligned}$$

for every $\gamma > 0$. Choosing $\gamma = 4C/c_-$, we conclude

$$\frac{1}{4} c_- \|\nabla^N \bar{h}_1^N(t)\|_{L^2}^2 \leq \frac{8C}{c_-} \|\tilde{k}^{N,\delta}(t)\|_{L^2}^2 + \frac{8C_a}{c_-} N^d |D_N|.$$

Applying the Poincaré inequality to the above, we also obtain the bound for $\|\bar{h}_1^N(t)\|_{L^2}^2$. \square

Since we now have the nice bounds for $\bar{h}_1^N(t)$ and $\bar{h}^N(t)$, we have one for $\bar{h}_2^N(t)$ also. We shall show the following proposition which says that the bound for $\bar{h}_2^N(t)$ implies the bound for $\langle k^{N,\delta}(t) \rangle$.

Proposition 3.12. *For $\alpha \in \mathbb{R}$, let $\bar{h}_{2,\alpha}^N(t)$ be the solution of*

$$\alpha = \operatorname{div}_N(A^N(t) \nabla^N \bar{h}_{2,\alpha}^N(t)) \quad (3.30)$$

with the Dirichlet boundary condition. We then have

$$\alpha^2 \leq C \|\nabla^N \bar{h}_{2,\alpha}^N\|_{L^2(D)}^2. \quad (3.31)$$

with a constant $C > 0$ independent of N, t and α .

Proof. Multiplying (3.30) by $h_{2,\alpha}^N$ and taking sum over D_N , we have

$$\alpha N^{-d} \sum_{x \in D_N} \bar{h}_{2,\alpha}^N(t, x/N) = N^{-d} \sum_{x \in D_N} \bar{h}_{2,\alpha}^N(t, x/N) \operatorname{div}_N (A^N(t) \nabla^N \bar{h}_{2,\alpha}^N(t, x/N)).$$

Performing the summation-by-parts at the right hand side, we obtain

$$\alpha N^{-d} \sum_{x \in D_N} \bar{h}_{2,\alpha}^N(t, x/N) = -N^{-d} \sum_{x \in \overline{D_N}} \nabla^N \bar{h}_{2,\alpha}^N(t, x/N) \cdot A^N(t) \nabla^N \bar{h}_{2,\alpha}^N(t, x/N). \quad (3.32)$$

We have used $\bar{h}_{2,\alpha}^N(t)$ satisfies the Dirichlet boundary condition. Noting that $h_{2,\alpha}^N$ can be expressed by

$$h_{2,\alpha}^N = \alpha h_{2,1}^N$$

because the right hand side of (3.30) is linear, we get

$$\begin{aligned} \left| \alpha^2 N^{-d} \sum_{x \in D_N} \bar{h}_{2,1}^N(t, x/N) \right| &= N^{-d} \sum_{x \in \overline{D_N}} \nabla^N \bar{h}_{2,\alpha}^N(t, x/N) \cdot A^N(t) \nabla^N \bar{h}_{2,\alpha}^N(t, x/N) \\ &\leq c_- N^{-d} \sum_{x \in \overline{D_N}} |\nabla^N \bar{h}_{2,\alpha}^N(t, x/N)|^2 \end{aligned}$$

from (3.25).

Here, once we have

$$\inf_N \left| N^{-d} \sum_{x \in D_N} \bar{h}_{2,1}^N(t, x/N) \right| > c \quad (3.33)$$

with a constant $c > 0$ independent of N and t , we immediately obtain the conclusion. We shall therefore show (3.33). Using (3.32) and (3.25), we get

$$\left| N^{-d} \sum_{x \in D_N} \bar{h}_{2,1}^N(t, x/N) \right| \geq c_- \|\nabla^N \bar{h}_{2,1}^N\|_{L^2(D)}^2. \quad (3.34)$$

For the function g^N introduced at the beginning of Section 3.5, we obtain

$$\begin{aligned} 1 &= N^{-d} \sum_{x \in D_N} g^N(x/N) \\ &= N^{-d} \sum_{x \in \overline{D_N}} \nabla^N g^N(x/N) \cdot A^N(t) \nabla^N \bar{h}_{2,1}^N(t, x/N) \\ &\leq C' \|\nabla^N g^N\|_{L^2(D)} \|\nabla^N \bar{h}_{2,1}^N(t)\|_{L^2(D)}. \end{aligned}$$

with a constant $C' > 0$ by using (3.25). Combining the above with (3.15) and (3.34), we get

$$\left| N^{-d} \sum_{x \in D_N} \bar{h}_{2,1}^N(t, x/N) \right| \geq \frac{c_-}{C'^2 c_g^2},$$

which shows (3.33). □

Once we obtain Proposition 3.11 and Proposition 3.12, we can easily show Proposition 3.10. Noting

$$\|\nabla^N h_2^N(t)\|_{L^2(D)}^2 \leq 2\|\nabla^N h^N(t)\|_{L^2(D)}^2 + 2\|\nabla^N h_1^N(t)\|_{L^2(D)}^2,$$

we have

$$\sup_N \sup_{0 \leq t \leq T} \|\nabla^N h_2^N(t)\|_{L^2(D)} < \infty$$

by using Proposition 3.7 and Proposition 3.11. On the other hand, since we have

$$|\langle k^{N,\delta}(t) \rangle|^2 \leq C \|\nabla^N h_2^N(t)\|_{L^2(D)}$$

from Proposition 3.12, we obtain (3.23) and therefore Proposition 3.10.

Corollary 3.13. *Under the assumption (3.21), we have the following bound:*

$$\sup_N \sup_{0 \leq t \leq T} (\|k^{N,\delta}(t)\|_{L^2(D)} + \|\nabla^N k^{N,\delta}(t)\|_{L^2(D)}) < \infty. \quad (3.35)$$

Proof. From Assmption 2.1 and the definition of $k^{N,\delta}$ as in Section 3.4, we can easily see

$$\|k^{N,\delta}(t)\|_{L^2(D)} \leq C_1 \sum_{x \in D_N} |k^{N,\delta}(t, x/N)|^2$$

and

$$\|\nabla^N k^{N,\delta}(t)\|_{L^2(D)}^2 \leq C_2 N^{-d} \sum_{x \in D_N} k^{N,\delta}(t, x/N) (-\Delta_N k^{N,\delta})(t, x/N)$$

with constants $C_1, C_2 > 0$ independent of N . These inequalities and Proposition 3.10 imply (3.35). \square

Once we have Corollary 3.13, we can also obtain uniform L^p -bound for $\nabla^N \bar{h}^N$. This uniform bound plays the key role in the derivation of PDE (1.3) from the height variable h^N .

Proposition 3.14. *We assume (3.21). We then have the following bounds:*

$$\sup_N \sup_{0 \leq t \leq T} \|k^{N,\delta}(t)\|_{L^p(D)}^p < \infty$$

and

$$\sup_N \sup_{0 \leq t \leq T} \|\nabla^N \bar{h}^{N,\delta}(t)\|_{L^p(D)}^p < \infty$$

for some $p > 2$.

Proof. Combining Corollary 3.13 with the similar argument to the proof of Proposition I.4 in [5], we obtain the first assertion. Applying the argument in Section 3.3 of [7] to (3.27), we also obtain the second assertion. \square

As an application of Proposition 3.14, we can obtain the following identity corresponding to the oscillation inequality in [2]. This also plays the key role in the derivation of PDE (1.3) from the height variable h^N .

Proposition 3.15. *We assume (3.21). For the solution $\bar{h}^{N,\delta}$ of (3.13) and $e \in \mathbb{Z}^d$ such that $|e| = 1$, we have*

$$\lim_{N \rightarrow \infty} N^{-d} \int_0^T \sum_{x \in \overline{D_N}} |\nabla^N \bar{h}^{N,\delta}(t, x/N + e/N) - \nabla^N \bar{h}^{N,\delta}(t, x/N)|^2 dt = 0. \quad (3.36)$$

Proof. Take $1 \leq i \leq d$ arbitrary. From (3.13), we can split

$$F_0^N(t) := N^{-d-2} \sum_{x \in \overline{D_N}} \nabla_i^N k^{N,\delta}(t, x/N) \nabla_i^N \bar{h}^{N,\delta}(t, x/N)$$

into three terms as follows:

$$\begin{aligned} F_0^N(t) &= N^{-d-2} \sum_{x \in \overline{D_N}} \sum_{j=1}^d \nabla_i^N \nabla \sigma(\nabla^N \bar{h}^{N,\delta}(t, x/N)) \nabla_i^N \nabla_j^N \bar{h}^{N,\delta}(t, x/N) \\ &\quad - N^{-d-2} \sum_{x \in \overline{D_N} \cap (D_N - e_i)^c} \nabla_i^N \operatorname{div}_N \nabla \sigma(\nabla^N \bar{h}^{N,\delta}(t, x/N)) \nabla_i^N \bar{h}^{N,\delta}(t, x/N) \\ &\quad + N^{-d-2} \sum_{x \in \overline{D_N} \cap (D_N - e_i)^c} \nabla_i^N k^{N,\delta}(t, x/N) \nabla_i^N \bar{h}^{N,\delta}(t, x/N) \\ &=: F_1^N(t) + F_2^N(t) + F_3^N(t) \end{aligned}$$

Here, we have used

$$\nabla^N \bar{h}^{N,\delta}(t, x/N) = 0, \quad x \in \mathbb{Z}^d \setminus \overline{D_N}.$$

Using (3.2), we have for $F_1^N(t)$

$$F_1^N(t) \leq -c_- N^{-d} \sum_{x \in \overline{D_N}} |\nabla^N \bar{h}^{N,\delta}(t, x/N + e_i/N) - \nabla^N \bar{h}^{N,\delta}(t, x/N)|^2,$$

which is nothing but our target. From now on, we shall show the remaining terms vanish when $N \rightarrow \infty$.

For $F_0^N(t)$, since we have

$$|F_0^N(t)| \leq 2N^{-2} \int_0^T \|\nabla^N k^{N,\delta}(t)\|_{L^2(D)}^2 dt + 2N^{-2} \int_0^T \|\nabla^N \bar{h}^{N,\delta}(t)\|_{L^2(D)}^2 dt,$$

by Schwarz's inequality, we obtain

$$\lim_{N \rightarrow \infty} \int_0^T |F_0^N(t)| dt = 0$$

from Proposition 3.7.

We shall next establish the bound for F_2^N and F_3^N . To do so, we shall at first make an L^2 bound for $\nabla^N \bar{h}^N$ on $B_N = \overline{D_N} \setminus D_N$. Choosing $r, q > 1$ such that $r = p/2, 1/r + 1/q = 1$ and applying Hölder's inequality, we obtain

$$\begin{aligned} N^{-d} \sum_{x \in \overline{B_N}} |\nabla^N \bar{h}^{N,\delta}(t, x/N)|^2 &\leq \frac{1}{r} \left(N^{-d} |\overline{B_N}| \right)^{r/2q} N^{-d} \sum_{x \in \overline{D_N}} |\nabla^N \bar{h}^{N,\delta}(t, x/N)|^p \\ &\quad + \frac{1}{q} \left(N^{-d} |\overline{B_N}| \right)^{1/2}. \end{aligned} \quad (3.37)$$

For $F_2^N(t)$, since we have

$$|F_2^N(t)| \leq N^{-d} \sum_{x \in \overline{B_N}} |\nabla^N \bar{h}^{N,\delta}(t, x/N)|^2,$$

we obtain

$$\lim_{N \rightarrow \infty} \int_0^T |F_2^N(t)| dt = 0$$

from Proposition 3.14. Also for $F_3^N(t)$, we have

$$|F_3^N(t)| \leq 2\alpha(N)N^{-d} \sum_{x \in \overline{B_N}} |\nabla^N \bar{h}^N(t, x/N)|^2 + 2\alpha(N)^{-1}N^{-d} \sum_{x \in \overline{D_N}} |\nabla^N k^{N,\delta}(t, x/N)|^2$$

for an arbitrary sequence $\{\alpha(N)\}$ of positive numbers. Choosing $\alpha(N) = N^\epsilon$ with $\epsilon > 0$ small enough, we obtain

$$\lim_{N \rightarrow \infty} \int_0^T |F_3^N(t)| dt = 0.$$

from Proposition 3.14. Summarizing above, we conclude (3.36). \square

3.7. Convergence of the solution for the discretized equation. In this subsection, let us show the solution $\bar{h}^{N,\delta}$ of the discretized equation (3.13) to the solution h^δ for the regularized equation (3.11), when the initial data is smooth enough. The goal is the following:

Theorem 3.16. *Assume $h_0 \in C_0^\infty(D)$. Then, the sequence of solutions $\{\bar{h}^{N,\delta}(t)\}$ for the discretized equation (3.13) with initial datum h_0^N converges as $N \rightarrow \infty$ to the unique solution $h^\delta(t)$ of (3.11) with initial data h_0 in the following sense:*

$$\lim_{N \rightarrow \infty} \|\bar{h}^{N,\delta}(t) - h^\delta(t)\|_{H^1(D)^*} = 0$$

holds for every $t > 0$.

Proof. To simplify notations, we omit the parameter δ when no confusion arises.

We shall at first show that we can take a subsequence $\{N'\}$ such that $\bar{h}^{N'}$ converges to the solution of (3.11). We arbitrarily choose $f \in C_0^\infty(D)$ such that

$$\int_D h_0(\theta) d\theta = \int_D f(\theta) d\theta$$

and define f^N by

$$f^N(\theta) = N^d \int_{B(x/N, 1/N)} f(\theta) d\theta, \quad \theta \in B(x/N, 1/N), \quad x \in \mathbb{Z}^d.$$

We introduce the polilinear interpolation used in [2], that is, \hat{h}^N is defined by follows:

$$\hat{h}^N(t, \theta) = \sum_{\alpha \in \{0,1\}^d} \left[\prod_{i=1}^d (\alpha_i \{N\theta_i\} + (1 - \alpha_i)(1 - \{N\theta_i\})) \right] \bar{h}^N \left(t, \frac{[N\theta] + \alpha}{N} \right), \quad (3.38)$$

where $[\cdot]$ and $\{\cdot\}$ denote the integral and the fractional parts, respectively. We also define \hat{k}^N by the similar manner. We then have

$$\sup_N \sup_{0 \leq t \leq T} \|\hat{h}^N(t)\|_{H^1(D)} < \infty$$

and

$$\sup_N \sup_{0 \leq t \leq T} \|\hat{k}^N(t)\|_{H^1(D)} < \infty$$

by Proposition 3.7 and Corollary 3.13. Using Proposition 3.7, Corollary 3.9, Corollary 3.13 and the bounds stated above, we can choose a subsequence $\{N'\}$ such that

$$\begin{aligned} \bar{h}^{N'} - f^N &\rightarrow \bar{g} \quad \text{strongly in } C([0, T], H), \\ \hat{h}^N &\rightarrow \hat{h} \quad \text{weakly in } L^2([0, T], H_0^1(D)), \\ \bar{k}^N &\rightarrow \bar{k} \quad \text{weakly in } L^2([0, T] \times D), \\ \hat{k}^N &\rightarrow \hat{k} \quad \text{weakly in } L^2([0, T], H^1(D)) \end{aligned}$$

as $N' \rightarrow \infty$ for some $\bar{g}, \hat{h}, \bar{k}, \hat{k}$. Letting $\bar{h} = \bar{g} + f$, we can easily see that $\bar{h} = \hat{h}$ and $\bar{k} = \hat{k}$. Furthermore, in this setting, $\|h^N\|_{-1, N}$ converges to $\|h\|_H$ as $N \rightarrow \infty$. Applying the argument in Step 3 of Proposition I.2 in [5], we obtain that the limit \bar{h} is the solution of (3.11) with initial data \bar{h}_0 . Furthermore, the uniqueness for (3.11) implies that the sequence $\{\bar{h}^N; N \geq 1\}$ itself converges to \bar{h} strongly in $C([0, T], H^1(D)^*)$, which shows the conclusion. \square

4. IDENTIFICATION OF EQUILIBRIUM STATES

In this section, let us study the structure of the equilibrium states for the dynamics on $(\mathbb{Z}^d)^*$ corresponding to (2.1). We will focus our attention to the relationship between stationarity and Gibbs property, since we have already known that the family of extremal canonical Gibbs measures coincides with the family of extremal grandcanonical Gibbs measures introduced by [5], see [6] for details.

4.1. Notations. In order to characterize the equilibrium states, we shall prepare several notations precisely. Note that we will follow the same manner as in [5] and [7].

Let $(\mathbb{Z}^d)^*$ be the set of all directed bonds $b = (x, y)$, $x, y \in \mathbb{Z}^d$, $|x - y| = 1$ in \mathbb{Z}^d . We write $x_b = x$ and $y_b = y$ for $b = (x, y)$. We denote the bond $(e_i, 0)$ by e_i again if it doesn't cause any confusion. For every subset Λ of \mathbb{Z}^d , we denote the set of all directed bonds included Λ and touching Λ by Λ^* and $\overline{\Lambda}^*$, respectively. That is,

$$\begin{aligned} \Lambda^* &:= \{b \in (\mathbb{Z}^d)^*; x_b \in \Lambda \text{ and } y_b \in \Lambda\}, \\ \overline{\Lambda}^* &:= \{b \in (\mathbb{Z}^d)^*; x_b \in \Lambda \text{ or } y_b \in \Lambda\}. \end{aligned}$$

For $\phi = \{\phi(x); x \in \mathbb{Z}^d\} \in \mathbb{R}^{\mathbb{Z}^d}$, the gradient ∇ is defined by

$$\nabla\phi(b) := \phi(x) - \phi(y), \quad b = (x, y) \in (\mathbb{Z}^d)^*.$$

Now, let \mathcal{X} be the family of all gradient fields $\eta \in \mathbb{R}^{(\mathbb{Z}^d)^*}$ which satisfy the plaquette condition (2.1) in [5], i.e., $\mathcal{X} = \{\eta \equiv \nabla\phi; \phi \in \mathbb{R}^{\mathbb{Z}^d}\}$. Let \mathbb{L}_r^2 be the set of all $\eta \in \mathbb{R}^{(\mathbb{Z}^d)^*}$

such that

$$|\eta|_r^2 := \sum_{b \in (\mathbb{Z}^d)^*} |\eta(b)|^2 e^{-2r|x_b|} < \infty.$$

We denote $\mathcal{X}_r = \mathcal{X} \cap \mathbb{L}_r^2$ equipped with the norm $|\cdot|_r$. We introduce the dynamics $\eta_t \in \mathcal{X}$ governed by the SDEs

$$d\eta_t(b) = -\nabla U(\eta_t)(b) dt + \sqrt{2d} \nabla \tilde{w}_t^{\mathbb{Z}^d}(b), \quad b \in (\mathbb{Z}^d)^*, \quad (4.1)$$

where $\{w_t^{\mathbb{Z}^d}(x); x \in \mathbb{Z}^d\}$ is the family of Gaussian processes with mean zero and covariance structure

$$E[\tilde{w}_t^{\mathbb{Z}^d}(x) \tilde{w}_s^{\mathbb{Z}^d}(y)] = -\Delta_{\mathbb{Z}^d}(x, y) t \wedge s, \quad x, y \in \mathbb{Z}^d, t, s \geq 0.$$

Since the coefficients are Lipschitz continuous in \mathcal{X}_r , this equation has the unique strong solution in \mathcal{X}_r for every $r > 0$. Note that $\eta_t := \nabla \phi_t$ defined from the solution ϕ_t of the SDE (2.1) on D_N satisfies (4.1) for $b \in \overline{D_N^*}$ and boundary conditions $\eta_t(b) = \nabla \psi^N(b)$ for $b \in (\mathbb{Z}^d)^* \setminus \overline{D_N^*}$, when replacing $\tilde{w}_t^{\mathbb{Z}^d}$ by \tilde{w}_t .

Let $\mathcal{P}(\mathcal{X})$ be the set of all probability measures on \mathcal{X} and let $\mathcal{P}_2(\mathcal{X})$ be those $\mu \in \mathcal{P}(\mathcal{X})$ satisfying $E^\mu[|\eta(b)|^2] < \infty$ for each $b \in (\mathbb{Z}^d)^*$. The measure $\mu \in \mathcal{P}_2(\mathcal{X})$ is sometimes called tempered. Let \mathcal{G} be the family of translation invariant, tempered Gibbs measures $\mu \in \mathcal{P}_2(\mathcal{X})$ introduced by [5], and \mathcal{G}_{ext} be the family of $\mu \in \mathcal{G}$ with ergodicity under spatial shifts. In the case with strict convexity of V , properties of Gibbs measures are studied quite well, see [5] and [2].

We define the differential operator ∂_x for $x \in \mathbb{Z}^d$ acting on $C_{\text{loc},b}^2(\mathbb{R}^{\mathbb{Z}^d})$ by

$$\partial_x = \frac{\partial}{\partial \phi(x)}.$$

We then have that the generator of (2.1) can be expressed by

$$\mathcal{L}_N = -4 \sum_{x \in D_N} \partial_x (\Delta_N \partial \cdot)(x) + 2 \sum_{x \in D_N} (\Delta_N U(\nabla \phi))(x) \partial_x.$$

We define the differential operator ∂_x for $x \in \mathbb{Z}^d$ acting on $C_{\text{loc}}^2(\mathcal{X})$ by

$$\partial_x := \sum_{b \in (\mathbb{Z}^d)^*; x_b = x} \frac{\partial}{\partial \eta(b)}$$

We define the differential operator \mathcal{L}_N acting on $C_{\text{loc}}^2(\mathcal{X})$ by

$$\mathcal{L}_N = -4 \sum_{x \in D_N} \partial_x (\Delta_N \partial \cdot)(x) + 2 \sum_{x \in D_N} (\Delta_N U(\eta))(x) \partial_x.$$

We then have that \mathcal{L}_N is the generator of $\eta_t = \nabla \phi_t$, where ϕ_t is the solution of (2.1).

We also define the differential operator $\mathcal{L}_{\mathbb{Z}^d}$ acting on $C_{\text{loc}}^2(\mathcal{X})$ by

$$\mathcal{L}_{\mathbb{Z}^d} = -4 \sum_{x \in \mathbb{Z}^d} \partial_x (\Delta \partial \cdot)(x) + 2 \sum_{x \in \mathbb{Z}^d} (\Delta U(\eta))(x) \partial_x.$$

To make notations keep simple, we simply denote $\mathcal{L}_{\mathbb{Z}^d}$ by \mathcal{L} if it does not cause any confusion. We also define

$$\mathcal{L}_\Lambda = -4 \sum_{x \in \Lambda} \partial_x (\Delta_\Lambda \partial \cdot)(x) + 2 \sum_{x \in \Lambda} (\Delta_\Lambda U(\eta))(x) \partial_x.$$

4.2. Relationship between stationary measures and Gibbs measures.

Theorem 4.1. *Let the probability measure μ on \mathcal{X} be translation invariant and tempered, that is,*

$$\sup_{b \in (\mathbb{Z}^d)^*} E^\mu[\eta(b)^2] < \infty$$

If μ satisfies

$$\int_{\mathcal{X}} \mathcal{L}F(\eta) d\mu = 0$$

for every $F \in C^2(\mathcal{X})$ with compact support, then μ is a canonical Gibbs measure introduced in [6].

The proof of Theorem 4.1 is similar to [3], which is based on [4].

We at first introduce $\Phi_\lambda : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\Phi_\lambda(u) = \frac{\lambda}{a} (1 + (\lambda u)^2)^{-m},$$

where

$$a = \int_{\mathbb{R}} (1 + u^2)^{-m} du.$$

For $\Lambda_n := [-n, n]^d \cap \mathbb{Z}^d$ we define $\Phi_n^\lambda : \mathcal{X}_{\Lambda_n^*} \rightarrow \mathbb{R}$ by

$$\Phi_n^\lambda(\eta) = \prod_{x \in \Lambda_n} \Phi_\lambda(\phi^{\eta,0}(x)),$$

where $\phi^{\eta,a}$ is the height variable satisfying $\nabla \phi^{\eta,a} = \eta$ and $\phi^{\eta,a}(0) = a$. Note that $\phi^{\eta,a}$ is uniquely determined by η and a . We also define $p_n^\lambda(\eta)$ by

$$p_n^\lambda(\eta) = \int \Phi_n^\lambda(\eta - \xi) \mu(d\xi).$$

Let $\Psi_n^\lambda(\eta, \xi) = \Phi_n^\lambda(\xi - \eta)$. Since $\Psi_n^\lambda(\cdot, \xi) \in C_{\text{loc}}^2(\mathcal{X})$, we have

$$\int \mathcal{L}\Psi_n^\lambda(\cdot, \xi)(\eta) \mu(d\eta) = 0.$$

Multiplying $F(\xi) \in C_{\text{loc}}^2(\mathcal{X})$ whose support is in Λ_n^* , and integrating in ξ by the uniform measure on $\mathcal{X}_{\Lambda_n^*}$, we obtain

$$\iint F(\xi) \mathcal{L}\Psi_n^\lambda(\cdot, \xi)(\eta) \mu(d\eta) d\xi_{\Lambda_n^*} = 0. \quad (4.2)$$

Applying Lemma 2.2 in [3] and the relationship

$$\begin{aligned} \frac{\partial \Psi_n^\lambda(\nabla \cdot, \nabla \psi)}{\partial \phi(x)}(\phi) &= -\frac{\partial \Phi_n^\lambda}{\partial \phi(x)}(\nabla \psi - \nabla \phi) = -\frac{\partial \Psi_n^\lambda(\nabla \phi, \nabla \cdot)}{\partial \psi(x)}(\psi) \\ \frac{\partial^2 \Psi_n^\lambda(\nabla \cdot, \nabla \psi)}{\partial \phi(x)^2}(\phi) &= \frac{\partial^2 \Phi_n^\lambda}{\partial \phi(x)^2}(\nabla \psi - \nabla \phi) = \frac{\partial^2 \Psi_n^\lambda(\nabla \phi, \nabla \cdot)}{\partial \psi(x)^2}(\psi) \end{aligned}$$

for $x \in \mathbb{Z}^d$ by the symmetricity of Φ^λ , the left hand side is calculated as follows:

$$\begin{aligned} & \iint F(\nabla\psi) \sum_{x \in \Lambda_n} \sum_{y \in \Lambda_n} (-\Delta)(x, y) \frac{\partial^2 \Psi_n^\lambda(\eta, \nabla \cdot)}{\partial \psi(x) \partial \psi(y)} \nu_{\Lambda_n, p}(d\psi) \mu(d\eta) \\ & + \iint F(\nabla\psi) \sum_{x \in \Lambda_n} \sum_{y \in \mathbb{Z}^d} (-\Delta)(x, y) U_y(\eta) \frac{\partial \Psi_n^\lambda(\eta, \nabla \cdot)}{\partial \psi(x)} \nu_{\Lambda_n, p}(d\psi) \mu(d\eta) \\ & =: I_1 + I_2, \end{aligned}$$

where $\nu_{\Lambda_n, p}$ is the measure on \mathbb{R}^{Λ_n} defined by

$$\nu_{\Lambda_n, p}(d\psi) = p(\psi(0)) \prod_{x \in \Lambda_n} d\psi(x)$$

with a probability density p on \mathbb{R} . Performing the integration-by-parts for I_1 , we have We shall first calculate I_1 . Performing integration-by-parts in ψ , we have

$$\begin{aligned} I_1 = & - \iint \sum_{x \in \Lambda_n} \sum_{y \in \Lambda_n} (-\Delta)(x, y) \frac{\partial F(\nabla \cdot)}{\partial \psi(x)} \frac{\partial \Psi_n^\lambda(\eta, \nabla \cdot)}{\partial \psi(y)} \nu_{\Lambda_n, p}(d\psi) \mu(d\eta) \\ & - \iint \sum_{y \in \Lambda_n} (-\Delta)(x, y) F(\nabla \psi) \frac{\partial \Psi_n^\lambda(\eta, \nabla \cdot)}{\partial \psi(0)} p'(\psi(0)) \prod_{x \in \Lambda_n} d\psi(x) \mu(d\eta). \end{aligned} \quad (4.3)$$

Noting that integrands of I_1 and the first term in the right hand side of (4.3) are function of $\nabla \psi$, each integral does not depend on the choice of p and therefore the second term does not also. Taking a sequence p_n such that $p'_n \rightarrow 0$ as $n \rightarrow \infty$, we conclude that the second term must be zero.

Let us choose F as

$$F(\nabla \psi) = f \left(\frac{p_n^\lambda(\nabla \psi)}{q_n(\nabla \psi)} \right), \quad (4.4)$$

with some bounded smooth function $f : \mathbb{R} \rightarrow \mathbb{R}$ and

$$q_n(\eta) = \exp(-H^\Lambda(\eta)), \quad \eta \in \mathcal{X}_{\Lambda_n^*},$$

where Z_n is the normalizing constant. Noting

$$\frac{\partial p_n^\lambda(\nabla \cdot)}{\partial \psi(x)} = \int \frac{\partial \Psi_n^\lambda(\eta, \nabla \cdot)}{\partial \psi(x)} \mu(d\eta)$$

and putting

$$r_n^\lambda(\eta) = \frac{p_n^\lambda(\eta)}{q_n(\eta)},$$

we have

$$\begin{aligned} I_1 = & - \sum_{x \in \Lambda_n} \sum_{y \in \Lambda_n} (-\Delta)(x, y) \int f' \left(r_n^\lambda(\nabla \psi) \right) \frac{\partial r_n^\lambda(\nabla \cdot)}{\partial \psi(x)} \frac{\partial r_n^\lambda(\nabla \cdot)}{\partial \psi(y)} q_n(\nabla \psi) \nu_{\Lambda_n, p}(d\psi) \\ & + \sum_{x \in \Lambda_n} \sum_{y \in \Lambda_n} (-\Delta)(x, y) \int f' \left(r_n^\lambda(\nabla \psi) \right) U_y^\Lambda(\nabla \psi) p_n^\lambda(\nabla \psi) \nu_{\Lambda_n, p}(d\psi). \end{aligned} \quad (4.5)$$

Next, we shall compute I_2 . Performing the integration-by-parts in $\psi(x)$ again, we have

$$\begin{aligned}
I_2 &= \sum_{x \in \Lambda_n} \sum_{y \in \mathbb{Z}^d \setminus \Lambda_n} (-\Delta)(x, y) \iint \frac{\partial F(\nabla \psi)}{\partial \psi(x)} U_y(\eta) \Psi_n^\lambda(\eta, \nabla \psi) \nu_{\Lambda_n, p}(d\psi) \mu(d\eta) \\
&\quad - \sum_{x \in \Lambda_n} \sum_{y \in \Lambda_n} (-\Delta)(x, y) \iint \frac{\partial F(\nabla \psi)}{\partial \psi(x)} (U_y(\eta) - U_y^\Lambda(\eta)) \Psi_n^\lambda(\eta, \nabla \psi) \nu_{\Lambda_n, p}(d\psi) \mu(d\eta) \\
&\quad - \sum_{x \in \Lambda_n} \sum_{y \in \Lambda_n} (-\Delta)(x, y) \iint \frac{\partial F(\nabla \psi)}{\partial \psi(x)} (U_y^\Lambda(\eta) - U_y^\Lambda(\nabla \psi)) \Psi_n^\lambda(\eta, \nabla \psi) \nu_{\Lambda_n, p}(d\psi) \mu(d\eta) \\
&\quad - \sum_{x \in \Lambda_n} \sum_{y \in \Lambda_n} (-\Delta)(x, y) \iint \frac{\partial F(\nabla \psi)}{\partial \psi(x)} U_y^\Lambda(\nabla \psi) p_n^\lambda(\nabla \psi) \nu_{\Lambda_n, p}(d\psi) \\
&=: R_1^\lambda(n, f) + R_2^\lambda(n, f) + R_3^\lambda(n, f) + R_4^\lambda(n, f).
\end{aligned} \tag{4.6}$$

Summarizing (4.3) and (4.6), we obtain

$$\begin{aligned}
F^\lambda(n, f) &:= \sum_{x \in \Lambda_n} \sum_{y \in \Lambda_n} (-\Delta)(x, y) \int f' (r_n^\lambda(\nabla \psi)) \frac{\partial r_n^\lambda(\nabla \cdot)}{\partial \psi(x)} \frac{\partial r_n^\lambda(\nabla \cdot)}{\partial \psi(y)} q_n(\nabla \psi) \nu_{\Lambda_n, p}(d\psi) \\
&= R_1^\lambda(n, f) + R_2^\lambda(n, f) + R_3^\lambda(n, f)
\end{aligned} \tag{4.7}$$

if we take F as in (4.4). Note that $F^\lambda(n, f)$ is finite when $f(x) = \log x$, by using $|\Delta(x, y)| \leq 2d$ and Lemma 2.3 of [3]. We denote $F^\lambda(n, f)$ with $f(x) = \log x$ simply by $F^\lambda(n)$.

From now on, we shall show that terms $R_i^\lambda(n, f)$ in the right hand side can be controled by $F^\lambda(n)$.

Lemma 4.2. *Assume that the function f satisfies $0 \leq uf'(u) \leq 1$ for every $u > 0$. We then have bounds for $R_1^\lambda(n, f)$, $R_2^\lambda(n, f)$ and $R_3^\lambda(n, f)$ in (4.6) as follows:*

$$|R_1^\lambda(n, f)| \leq KC(n)^{1/2} F^\lambda(n)^{1/2} \tag{4.8}$$

$$|R_2^\lambda(n, f)| \leq K\lambda^{-1} F^\lambda(n)^{1/2} \tag{4.9}$$

$$|R_3^\lambda(n, f)| \leq KC(n)^{1/2} F^\lambda(n)^{1/2} \tag{4.10}$$

with a constant $K > 0$ independent in n and λ , where $C_x(n)$ is defined by

$$\begin{aligned}
C(n) &= \sum_{x \in \Lambda_{n+1}} \sum_{b \in (\mathbb{Z}^d)^* \setminus \Lambda^* : x_b = x} \int c_b^2(\xi, n, \mu) \mu(d\xi), \\
c_b(\xi, n, \mu) &= \int V'(\eta(b)) \mu(d\eta | \mathcal{F}_{\Lambda_n^*})(\xi).
\end{aligned}$$

Proof. We at first obtain

$$\begin{aligned}
& \iint f' \left(\frac{p_n^\lambda}{q_n} \right)^2 \sum_{x \in \Lambda_n} \sum_{y \in \Lambda_n} (-\Delta)(x, y) \frac{\partial r_n^\lambda}{\partial \psi(x)} \frac{\partial r_n^\lambda}{\partial \psi(y)} \Psi_n^\lambda(\eta, \nabla \psi) \nu_{\Lambda_n, p}(d\psi) \mu(d\eta) \\
& \leq \iint \left(\frac{p_n^\lambda}{q_n} \right)^{-2} \sum_{x \in \Lambda_n} \sum_{y \in \Lambda_n} (-\Delta)(x, y) \frac{\partial r_n^\lambda}{\partial \psi(x)} \frac{\partial r_n^\lambda}{\partial \psi(y)} p_n^\lambda(\nabla \psi) \nu_{\Lambda_n, p}(d\psi) \\
& = \int \left(\frac{p_n^\lambda}{q_n} \right)^{-1} \sum_{x \in \Lambda_n} \sum_{y \in \Lambda_n} (-\Delta)(x, y) \frac{\partial r_n^\lambda}{\partial \psi(x)} \frac{\partial r_n^\lambda}{\partial \psi(y)} q_n(\nabla \psi) \nu_{\Lambda_n, p}(d\psi) = F^\lambda(n)
\end{aligned}$$

by assumption on f . Note that $(-\Delta)$ is nonnegative definite, indeed, we have

$$\sum_{x \in \Lambda_m} (-\Delta)(x, y) \phi(x) \phi(y) \geq 0$$

for every $m \geq 1$ and $\phi \in \mathbb{R}^{\mathbb{Z}^d}$ such that $\phi(x) = 0$ on Λ_m^c . Furthermore, we have the Schwarz inequality of the following form:

$$\begin{aligned}
& \left| \sum_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} (-\Delta)(x, y) \phi(x) \psi(y) \right| \\
& \leq \left(\sum_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} (-\Delta)(x, y) \phi(x) \phi(y) \right)^{1/2} \left(\sum_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} (-\Delta)(x, y) \psi(x) \psi(y) \right)^{1/2} \quad (4.11)
\end{aligned}$$

for every $\phi, \psi \in \mathbb{R}^{\mathbb{Z}^d}$ with $\phi(x) = \psi(x) = 0$ on Λ_m^c for some $m \geq 1$. Using the above, we obtain

$$|R_1^\lambda(n, f)| \leq C F^\lambda(n)^{1/2} C(n)^{1/2}$$

for some constant $C > 0$, which shows (4.8). We can also obtain (4.9) and (4.10) by the similar argument to the above. Note that (2.12) and (2.13) in [3] is required for the proof of (4.9). \square

Let us continue the calculation for $F^\lambda(n, f)$. Summarizing (4.7)-(4.10), we get

$$F^\lambda(n, f) \leq \frac{1}{2} F^\lambda(n) + K'(C(n) + \lambda^{-1} n^d)$$

with a constant $K' > 0$. Using Fatou's lemma, we conclude

$$\frac{1}{2} F^\lambda(n) \leq K'(C(n) + \lambda^{-1} n^d). \quad (4.12)$$

We shall next give a lower bound for the left hand side of (4.12). For $\ell \in \mathbb{N}$, let us take $\tilde{\Lambda} \subset \Lambda_n$ by

$$\tilde{\Lambda} = \bigcup_{x \in \mathcal{A}_{n, \ell}} \Lambda_\ell(x),$$

where $\Lambda_\ell(x) = \Lambda_\ell + x$ and

$$\mathcal{A}_{n, \ell} = \{x \in ((2\ell + 3)\mathbb{Z})^d; \Lambda_\ell(x) \subset \Lambda_{n-1}\}.$$

Because boxes $\Lambda_\ell(x)$ appearing above are disjoint, we get

$$\begin{aligned}
F^\lambda(n) &= \frac{1}{2} \int \sum_{x \in \Lambda_n} \sum_{y \in \mathbb{Z}^d; |x-y|=1} \left(\frac{\partial \sqrt{r_n^\lambda}}{\partial \psi(x)} - \frac{\partial \sqrt{r_n^\lambda}}{\partial \psi(y)} \right)^2 q_n(\nabla \psi) \nu_{\Lambda_n, p}(d\psi) \\
&\geq \frac{1}{2} \int \sum_{z \in \mathcal{A}_{n, \ell}} \sum_{x \in \Lambda_\ell(z)} \sum_{y \in \Lambda_\ell(z); |x-y|=1} \left(\frac{\partial \sqrt{r_n^\lambda}}{\partial \psi(x)} - \frac{\partial \sqrt{r_n^\lambda}}{\partial \psi(y)} \right)^2 q_n(\nabla \psi) \nu_{\Lambda_n, p}(d\psi) \\
&= \frac{1}{2} \sum_{z \in \mathcal{A}_{n, \ell}} I^{\Lambda_\ell(x)}(\mu_n^\lambda),
\end{aligned}$$

where I^Λ is the entropy production rate defined by

$$I^\Lambda(\tilde{\mu}) := \sup \left\{ \int \frac{-\mathcal{L}^\Lambda u}{u} d\tilde{\mu}; u \in C_b^2(\mathcal{X}), \mathcal{F}_{\Lambda^*}\text{-measurable}, u \geq 1 \right\}$$

for a finite $\Lambda \subset \mathbb{Z}^d$ and $\tilde{\mu}$ on \mathcal{X} or \mathcal{X}_{Λ^*} with n large enough. Applying (4.12) and taking the limit $\lambda \rightarrow \infty$, we have

$$\frac{1}{2} \sum_{z \in \mathcal{A}_{n, \ell}} I^{\Lambda_\ell(x)}(\mu) \leq 4K'C(n)$$

by the lower semicontinuity of the entropy production rate. Since $I^{\Lambda_\ell(x)}(\mu_n)$ does not depend on x by the translation invariance of μ and $C(n) = O(n^{d-1})$, we get

$$I^{\Lambda_\ell}(\mu) \leq C\ell^d n^{-1} \quad (4.13)$$

with a constant $C > 0$ independent of n . Taking the limit $n \rightarrow \infty$, we finally conclude that

$$I^{\Lambda_\ell}(\mu) = 0 \quad (4.14)$$

for every $\ell \in \mathbb{N}$. Repeating the same argument as in the proof of Lemma 5.2 in [6], we obtain that μ is a canonical Gibbs measure introduced in [6].

5. PROOF OF THEOREM 2.1

In this section, we shall give the proof of our main result, Theorem 2.1. Before that, we shall prepare several bounds, which play key role in the proof.

5.1. A priori bounds for stochastic processes. In this section, the goal is to prove following proposition:

Proposition 5.1. *There exist constants C_1, C_2 independent of N such that*

$$E\|h^N(t)\|_{-1, N}^2 + N^{-d} E \int_0^t \sum_{x \in D_N^*} (\nabla \phi_s^N(x))^2 ds \leq C_1 E\|h^N(0)\|_{-1, N}^2 + C_2(1+t)$$

holds for every $t \geq 0$.

Proof. Let us use the function \tilde{f}^N and $\tilde{\psi}^N$ defined in Section 2.1. We define ψ^N by

$$\psi^N(x) = \langle \phi_0^N \rangle \tilde{\psi}^N(x),$$

where $\langle \phi_0^N \rangle$ is defined by

$$\langle \phi_0^N \rangle = N^{-d-1} \sum_{y \in D_N} \phi_0^N(y).$$

We denote the macroscopic height variable associated with the microscopic height variable ψ^N by f^N , that is,

$$f^N(\theta) = \sum_{x \in \mathbb{Z}^d} N^{-1} \psi^N(x) 1_{B(x/N, 1/N)}(\theta), \quad \theta \in \mathbb{R}^d.$$

Calculating $\|h^N(t) - f^N\|_{-1,N}^2$ by Itô's formula, we obtain

$$\begin{aligned} d\|h^N(t) - f^N\|_{-1,N}^2 &= -2N^{-d} \sum_{b \in \overline{D_N^*}} (\nabla \phi_t^N(b) - \nabla \psi^N(b)) V'(\nabla \phi_t^N(b)) dt \\ &\quad + 2N^{-d} \sum_{x \in D_N} (-\Delta_{D_N})^{-1} (\phi_t^N - \psi^N)(x) d\tilde{w}_t(y) \\ &\quad + 2N^{-d} |D_N| dt. \end{aligned}$$

Therefore, integrating in t and taking the expectation, we get

$$\begin{aligned} E\|h^N(T) - f^N\|_{-1,N}^2 &= E\|h^N(0) - f^N\|_{-1,N}^2 \\ &\quad - 2E \int_0^T N^{-d} \sum_{b \in \overline{D_N^*}} \nabla \phi_t^N(b) V'(\nabla \phi_t^N(b)) dt \\ &\quad + 2E \int_0^T N^{-d} \sum_{b \in \overline{D_N^*}} \nabla \psi^N(b) V'(\nabla \phi_t^N(b)) dt \\ &\quad + 2N^{-d} |D_N| T. \end{aligned} \tag{5.1}$$

Applying (3.15) to the third term in the right hand side, we get

$$\begin{aligned} \left| 2N^{-d} \sum_{b \in \overline{D_N^*}} \nabla \psi^N(b) V'(\nabla \phi_t^N(b)) \right| &\leq 2c_f |\langle \phi_0^N \rangle| \left(N^{-d} \sum_{b \in \overline{D_N^*}} (V'(\nabla \phi_t^N(b)))^2 \right)^{1/2} \\ &\leq 2c_f |\langle \phi_0^N \rangle| c_+ \left(N^{-d} \sum_{b \in \overline{D_N^*}} (\nabla \phi_t^N(b))^2 \right)^{1/2} \\ &\leq 2\gamma N^{-d} \sum_{b \in \overline{D_N^*}} (\nabla \phi_t^N(b))^2 + 8\gamma^{-1} c_f^2 |\langle \phi_0^N \rangle|^2 c_+^2 \end{aligned}$$

for every $\gamma > 0$. Plugging the above with $\gamma = c_-/2$ into (5.1), we finally obtain

$$\begin{aligned} E\|h^N(T) - f^N\|_{-1,N}^2 &\leq E\|h^N(0) - f^N\|_{-1,N}^2 \\ &\quad - c_-E \int_0^T N^{-d} \sum_{b \in \overline{D_N^*}} (\nabla \phi_t^N(b))^2 dt \\ &\quad + 2N^{-d}|D_N|T + 16c_-^{-1}c_+^2|\langle \phi_0^N \rangle|^2 T. \end{aligned}$$

Noting $|\langle \phi_0^N \rangle|^2 \leq \|h^N(0)\|_{-1,N}^2$, we obtain the conclusion. \square

5.2. Coupled local equilibria. In this subsection, we shall introduce the coupled measure and identify its limit point, as in [5] and [7]. Let us denote the (discrete) gradient of $\bar{h}^{N,\delta}$ introduced by (3.13) by $u_s^{N,\delta}$, that is,

$$u_s^{N,\delta}(x) = \nabla^N \bar{h}^{N,\delta}(s, x/N), \quad x \in \mathbb{Z}^d,$$

and the law of $\nabla \phi_s^N$ on \mathcal{X} by μ_s^N . Using these notations, we introduce the coupled measure $p^N(d\eta du)$ on $\mathcal{X} \times \mathbb{R}^d$ by

$$p^{N,\delta}(d\eta du) = t^{-1}|\overline{D_N}|^{-1} \int_0^t \sum_{x \in \overline{D_N}} (\mu_s^N \circ \tau_x^{-1})(d\eta) \delta_{u_s^{N,\delta}(x)}(du) ds, \quad (5.2)$$

where τ_x is the spatial shift by $x \in \mathbb{Z}^d$. We note that the sequence $\{p^N\}$ is tight as the probability measures on $\mathcal{X} \times \mathbb{R}^d$ since we have Proposition 5.1 and Proposition 3.7.

Proposition 5.2. *For every limit point p of $\{p^{N,\delta}\}$, there exists a probability measure $\lambda(du dv)$ on $\mathbb{R}^d \times \mathbb{R}^d$ such that the relationship*

$$p(d\eta du) = \int_{\mathbb{R}^d} \mu_v(d\eta) \lambda(du dv)$$

holds with ergodic Gibbs measures $\{\mu_u; u \in \mathbb{R}^d\}$ introduced by [5].

Proof. To keep notation simple, let us omit the parameter δ when no confusion arises.

For $\varphi \in C_b^1(\mathbb{R}^d)$, we define the signed measure $p^N(d\eta, \varphi)$ on \mathcal{X} by

$$p^N(d\eta, \varphi) = \int_{\mathbb{R}^d} p^N(d\eta du) \varphi(u).$$

It is sufficient to show that every limit point $p(d\eta, \varphi)$ of $\{p^N(d\eta, \varphi)\}$ is translation invariant and satisfies

$$\int_{\mathcal{X}} \mathcal{L}F(\eta) p(d\eta, \varphi) = 0 \quad (5.3)$$

for every $F \in C_b^2(\mathcal{X})$ with a compact support, see Theorem 4.1 of [5].

We shall at first show the limit point $p(d\eta, \varphi)$ is translation invariant. For $F \in C_b^2(\mathcal{X})$ and $e \in \mathbb{Z}^d$ such that $|e| = 1$, we have

$$\begin{aligned} & \left| \int_{\mathcal{X}} F(\eta) p^N(d\eta, \varphi) - \int_{\mathcal{X}} F(\eta) p^N(\cdot, \varphi) \circ \tau_e(d\eta) \right| \\ & \leq t^{-1} |\overline{D_N}|^{-1} \|F\|_{\infty} \|\varphi'\|_{\infty} \int_0^t \sum_{x \in \overline{\overline{D_N}}} |u^N(s, x/N) - u^N(s, x/N - e/N)| ds \\ & \quad + |\overline{D_N}|^{-1} \|F\|_{\infty} \|\varphi\|_{\infty} \left(\left| \overline{D_N} \cap (\overline{D_N} + e)^c \right| + \left| (\overline{D_N} + e) \cap \overline{D_N}^c \right| \right). \end{aligned}$$

Since the first term vanishes as $N \rightarrow \infty$ by Proposition 3.15, we obtain

$$\lim_{N \rightarrow \infty} \left| \int_{\mathcal{X}} F(\eta) p^N(d\eta, \varphi) - \int_{\mathcal{X}} F(\eta) p^N(\cdot, \varphi) \circ \tau_e(d\eta) \right| = 0,$$

which shows that the limit $p(d\eta, \varphi)$ is translation invariant.

Let us show (5.3). Fix $F \in C_b^2(\mathcal{X})$ with compact support, and take $L \in \mathbb{N}$ such that $\text{supp}(F) \subset \Lambda_L^*$, where $\Lambda_L = [-L, L]^d \cap \mathbb{Z}^d$. We then obtain for N large enough

$$\begin{aligned} \int_{\mathcal{X}} \mathcal{L}F(\eta) p^N(d\eta, \varphi) &= t^{-1} |D_N|^{-1} \int_0^t \sum_{x \in \overline{D_N}} \varphi(u_s^N(x)) E^{\mu_s^N \circ \tau_x} [\mathcal{L}F(\eta)] ds \\ &= t^{-1} |D_N|^{-1} \int_0^t \sum_{x \in \mathcal{A}_{N,L}} \varphi(u_s^N(x)) E^{\mu_s^N} [\mathcal{L}_{D_N}(F \circ \tau_x^{-1})(\eta)] ds \\ & \quad + t^{-1} |D_N|^{-1} \int_0^t \sum_{x \in \overline{D_N} \setminus \mathcal{A}_{N,L}} \varphi(u_s^N(x)) E^{\mu_s^N} [\mathcal{L}_{D_N}(F \circ \tau_x^{-1})(\eta)] ds \\ & \quad + t^{-1} |D_N|^{-1} \int_0^t \sum_{x \in \overline{D_N}} \varphi(u_s^N(x)) E^{\mu_s^N \circ \tau_x} [\mathcal{L}_{\mathbb{Z}^d \setminus D_N} F(\eta)] ds \\ &=: F_1^N + F_2^N + F_3^N, \end{aligned}$$

where the set $\mathcal{A}_{N,L}$ is defined by

$$\mathcal{A}_{N,L} = \overline{D_N} \cap \left(\bigcup_{z \in \partial D_N} (\Lambda_{2L} + z) \right)^c.$$

Since $\text{supp}(F \circ \tau_x^{-1}) \subset \overline{D_N}$ if $x \in \mathcal{A}_{N,L}$, we obtain

$$\begin{aligned} F_1^N &= t^{-1} |D_N|^{-1} N^{-4} \int_0^t \sum_{x \in \mathcal{A}_{N,L}} \varphi(u_s^N(x)) E^{\mu_s^N} [(F \circ \tau_x^{-1})(\nabla \phi_t^N)] \\ & \quad - t^{-1} |D_N|^{-1} N^{-4} \int_0^t \sum_{x \in \mathcal{A}_{N,L}} \varphi(u_s^N(x)) E^{\mu_s^N} [(F \circ \tau_x^{-1})(\nabla \phi_0^N)] \\ & \quad - t^{-1} |D_N|^{-1} N^{-4} \int_0^t \sum_{x \in \mathcal{A}_{N,L}} \frac{\partial}{\partial s} \varphi(u_s^N(x)) E^{\mu_s^N} [(F \circ \tau_x^{-1})(\nabla \phi_s^N)] ds \end{aligned}$$

by Itô's formula. Since $F \in C_b^2(\mathcal{X})$, we obtain

$$|F_1^N| \leq C_1 N^{-4} + C_2 N^{-4} \int_0^t \left\| \frac{\partial}{\partial s} \varphi(u_s^N(x)) \right\|_{L^1} ds$$

with some constants $C_1, C_2 > 0$ independent of N . Combining the above with Proposition 3.8, we have

$$\lim_{N \rightarrow \infty} |F_1^N| = 0.$$

For F_2^N , we have

$$\begin{aligned} |F_2^N| &\leq t^{-1} \|\varphi\|_\infty |D_N|^{-1} \int_0^t \sum_{x \in \overline{D_N} \setminus \mathcal{A}_{N,L}} E^{\mu_s^N} [|\mathcal{L}_{D_N}(F \circ \tau_x^{-1})(\eta)|] ds \\ &\leq t^{-1} \|\varphi\|_\infty |D_N|^{-1} \int_0^t \sum_{x \in \overline{D_N} \setminus \mathcal{A}_{N,L}} \sum_{y \in \Lambda_L + x} E^{\mu_s^N} [|\mathcal{L}_y(F \circ \tau_x^{-1})(\eta)|] ds \\ &\leq C_3 |D_N|^{-1} |\overline{D_N} \setminus \mathcal{A}_{N,L}| \\ &\quad + C_4 |D_N|^{-1} \int_0^t \sum_{x \in \overline{D_N} \setminus \mathcal{A}_{N,L}} \sum_{b \in (\mathbb{Z}^d)^*: x_b \in \Lambda_{2L} + x} E^{\mu_s^N} [|\nabla \phi_s^N(b)|] ds \\ &=: F_{2,1}^N + F_{2,2}^N \end{aligned}$$

with some constants $C_3, C_4 > 0$ independent of N . We can easily see that

$$\lim_{N \rightarrow \infty} F_{2,1}^N = 0.$$

For $F_{2,2}^N$, applying Schwarz's inequality, we obtain

$$F_{2,2}^N \leq C_5 N^{-1/2} |D_N|^{-1} E \int_0^t \sum_{b \in \overline{D_N}^*} |\nabla \phi_s^N(b)|^2 ds + C_6 N^{1/2} |D_N|^{-1} |\partial D_N|$$

with some constants $C_5, C_6 > 0$ independent of N , which indicates

$$\lim_{N \rightarrow \infty} F_{2,2}^N = 0$$

from Proposition 5.1. We therefore conclude

$$\lim_{N \rightarrow \infty} F_2^N = 0.$$

Since we obtain

$$F_3^N \leq C_7 N^{-1/2} |D_N|^{-1} E \int_0^t \sum_{b \in \overline{D_N}^*} |\nabla \phi_s^N(b)|^2 ds + C_8 N^{1/2} |D_N|^{-1} |\partial D_N|,$$

with some constants $C_7, C_8 > 0$ independent of N by a similar argument to that for F_2^N , we obtain

$$\lim_{N \rightarrow \infty} F_3^N = 0.$$

Summarizing the above, the identity (5.3) is concluded. \square

5.3. Derivation of the macroscopic equation under (3.21). Let us first prove Theorem 2.1 under the following assumption: $h^N(0)$ is given by $h^N(0) = \bar{h}_0^N$, where \bar{h}_0^N is defined by (3.14) with $h_0 \in C_0^\infty(D)$. Since we have

$$\begin{aligned} \|h^N(t) - h(t)\|_{H^1(D)^*} &\leq \|h^N(t) - \bar{h}^{N,\delta}(t)\|_{H^1(D)^*} \\ &\quad + \|\bar{h}^{N,\delta}(t) - h^\delta(t)\|_{H^1(D)^*} \\ &\quad + \|h^\delta(t) - h(t)\|_{H^1(D)^*}, \end{aligned}$$

it is sufficient for our goal to show

$$\limsup_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} E \|h^N(t) - \bar{h}^{N,\delta}(t)\|_{-1,N}^2 = 0, \quad (5.4)$$

by using (3.1). Using Itô's formula, we obtain

$$\begin{aligned} E \|h^N(t) - \bar{h}^{N,\delta}(t)\|_{-1,N}^2 &= -tN^{-d} \left| \overline{D_N} \right| \sum_{i=1}^d \int_{\mathcal{X} \times \mathbb{R}^d} \{ \eta(e_i) V'(\eta(e_i)) - u_i V'(\eta(e_i)) - \eta(e_i) \nabla_i \sigma^\delta(u) \\ &\quad + u_i \nabla_i \sigma^\delta(u) - 1 \} p^{N,\delta}(d\eta du) + tN^{-d} |\partial D_N|, \end{aligned}$$

where $p^{N,\delta}$ is the coupled measure introduced by (5.2). Applyin the same argument as in the Section 6 of [5] with Propositions 3.14 and 5.1, we conclude conclude (5.4).

5.4. Derivation of the macroscopic equation in general cases. Let us remove the assumption imposed at Section 5.3 and complete the proof of Theorem 2.1. For this aim, we prepare the following lemma:

Lemma 5.3. *Let ϕ_t and $\tilde{\phi}_t$ be the solution of (2.1) with common Gaussian processes $\{\tilde{w}_t(x); x \in D_N\}$ and let h^N and \tilde{h}^N be the macroscopic height variables corresponding to ϕ_t and $\tilde{\phi}_t$, respectively. Then, for every $t > 0$ and $N \geq 1$*

$$E \|h^N(t) - \tilde{h}^N(t)\|_{-1,N}^2 \leq E \|h^N(0) - \tilde{h}^N(0)\|_{-1,N}^2$$

holds.

Noting that ϕ_t and $\tilde{\phi}_t$ satisfy the same boundary condition, the quite same argument as in the proof of Proposition 5.1 can be applicable. We therefore omit the proof.

We shall approximate $h_0 \in L^2(D)$ by $h_0^\epsilon \in C_0^2(D)$ in the sense of

$$\lim_{\epsilon \rightarrow 0} \|h_0^\epsilon - h_0\|_{H^1(D)^*} = 0.$$

Let ϕ_t^ϵ be the solution of (2.1) with common Gaussian processes $\{\tilde{w}_t(x); x \in D_N\}$ as ϕ_t and with initial data ϕ_0^ϵ which are defined by

$$\phi_0^\epsilon(x) = \begin{cases} N^{d+1} \int_{B(x/N, 1/N)} h_0^\epsilon(\theta), & x \in D_N, \\ 0, & x \in \mathbb{Z}^d \setminus D_N. \end{cases}$$

Letting $h^{N,\epsilon}$ be the macroscopic height variable corresponding to ϕ_t^ϵ and h^ϵ be the solution of (1.3) with the initial data h_0^ϵ , we then have

$$\begin{aligned} E\|h^N(t) - h(t)\|_{H^1(D)^*}^2 &\leq 4E\|h^N(t) - h^{N,\epsilon}(t)\|_{H^1(D)^*}^2 + 4E\|h^{N,\epsilon}(t) - h^\epsilon(t)\|_{H^1(D)^*}^2 \\ &\quad + 8\|h^\epsilon(t) - h(t)\|_{H^1(D)^*}^2 \\ &\leq 4E\|h^N(t) - h^{N,\epsilon}(t)\|_{-1,N}^2 + 4E\|h^{N,\epsilon}(t) - h^\epsilon(t)\|_{H^1(D)^*}^2 \\ &\quad + 8\|h^\epsilon(t) - h(t)\|_{H^1(D)^*}^2 \end{aligned}$$

by (3.1). Here, applying the result proved in the previous subsection, Lemma 5.3 and Proposition 3.5, we conclude Theorem 2.1 in general settings.

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